What is a …

toric variety?

January 7, 2021

slides & more information available at alicia.lamarche.xyz/talks/whatisatoricvariety.pdf

Also, anything in blue is a link!

acknowledgements

- **MSRI Summer Schools** - 2019 Session on Toric Varieties run by David Cox & Hal Schenck
- "***Toric Varieties***" by Cox, Little, and Schenck
- "**What is a Toric Variety?**" by David Cox
- "***What is… a Toric Variety***?" by Erza Miller
- ***Matt Ballard*** (UofSC), ***Alex Duncan*** (UofSC), ***Patrick McFaddin*** (Fordham), Lenny Jones (Shippensburg University)

Goal:

Explore (normal) toric varieties and their associated *fans* by constructing the fan for the complex projective plane.

“Toric varieties form an important and rich class of examples in algebraic geometry, which often provide a testing ground for theorems. The geometry of a toric variety is fully determined by the combinatorics of its associated fan, which often makes computations far more tractable.”

–*[Wikipedia](https://en.wikipedia.org)*

Classical *Algebraic Geometry* aims to answer questions about the solution sets of systems of polynomials.

These are called <u>varieties</u>

Classical *Algebraic Geometry* aims to answer questions about the solution sets of systems of polynomials.

algebra

solving for x in equation

$$ax^2+bx+c=0$$

Geometry

Gives points in $\mathbb{R}^2$ where the parabola given by ax^2+bx+c intersects the x-axis

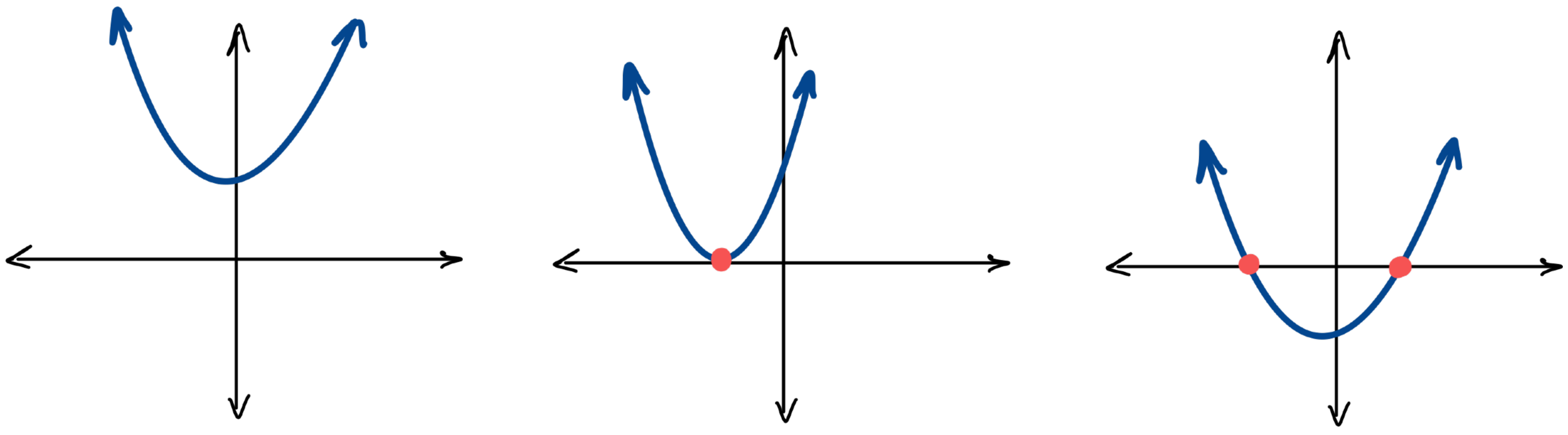
Classical *Algebraic Geometry* aims to answer questions about the solution sets of systems of polynomials.

algebra

Geometry

The discriminant: $b^2 - 4ac$

Sign tells us how many times the parabola given by $ax^2 + bx + c$ intersects the x-axis



$b^2 - 4ac < 0$ $\quad$ $b^2 - 4ac = 0$ $\quad$ $b^2 - 4ac > 0$

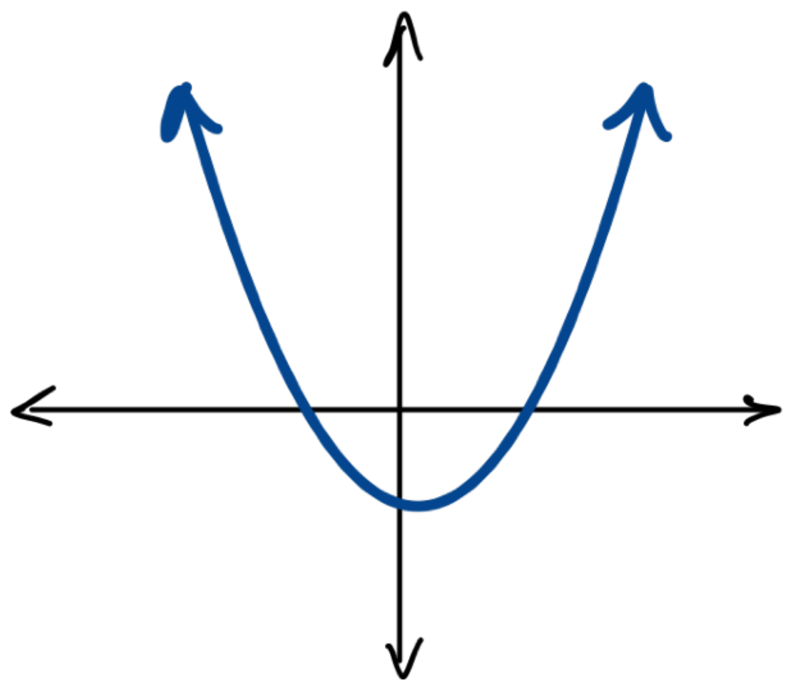
Classical *Algebraic Geometry* aims to answer questions about the solution sets of systems of polynomials.

algebra

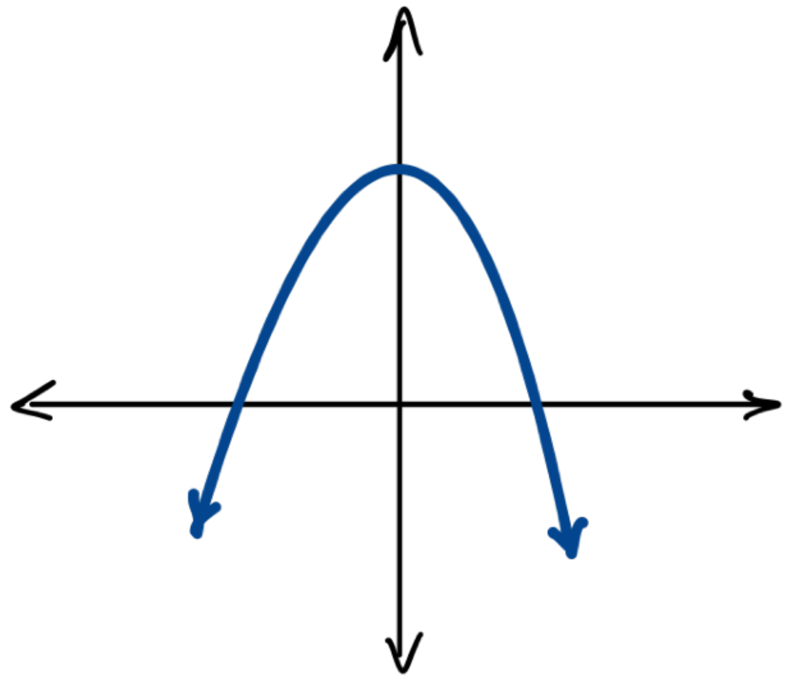
Geometry

The leading coefficient "a"

sign tells us if parabola opens upwards or downwards (concavity)



$a > 0$ — concave up (opens upwards)



$a < 0$ - concave down (opens down)

Slightly less classical *Algebraic Geometry* (à la ***Hartshorne***) gives us a more rigorous framework to play with.

Definition. Given a field k and positive integer n, we define the n-dimensional *affine space* over k to be the set

$$k^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_k \in k\}$$

Definition. A *polynomial* $f \in k[x_1, \ldots, x_n]$ can be thought of as a function from k^n to k

Ex. Consider $f(x,y) = x^2 + y$ in $\mathbb{R}[x,y]$.

I can take any point (a_1, a_2) in $\mathbb{R}^2$ and evaluate $f(x,y)$ at this point

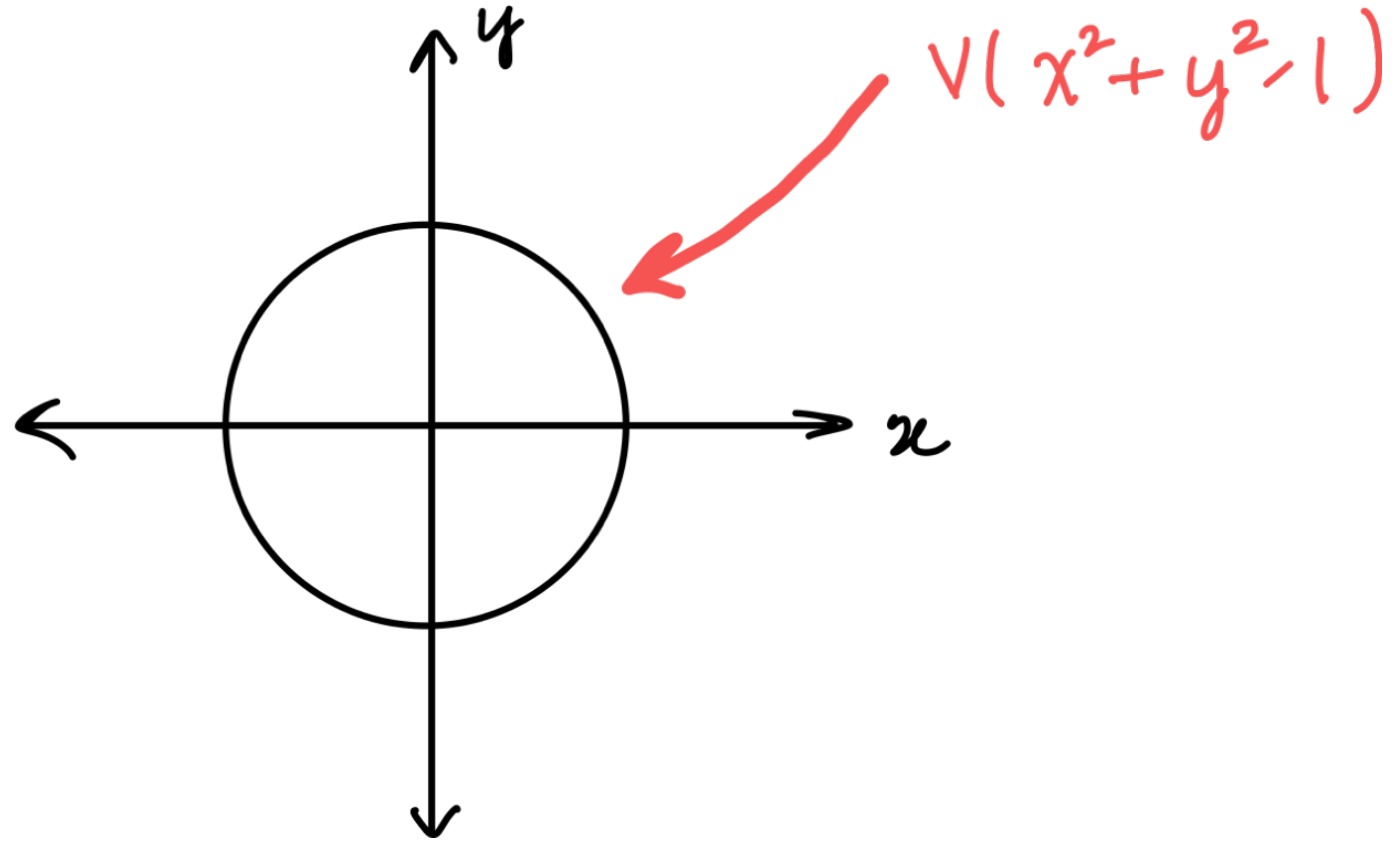
$f(a_1, a_2) = a_1^2 + a_2$ to get an element of $\mathbb{R}$.

Definition. Given a field k and polynomials $f_1, \ldots, f_s$ in $k[x_1, \ldots, x_n]$, define:

$$V(f_1, \ldots, f_s) = \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

We'll call $V(f_1, \ldots, f_s)$ the *affine variety* defined by $f_1, \ldots, f_s$.

Ex. Consider $V(x^2+y^2-1)$ sitting inside of $\mathbb{R}^2$.



Definition. Given a field k and polynomials $f_1, \ldots, f_s$ in $k[x_1, \ldots, x_n]$, define:

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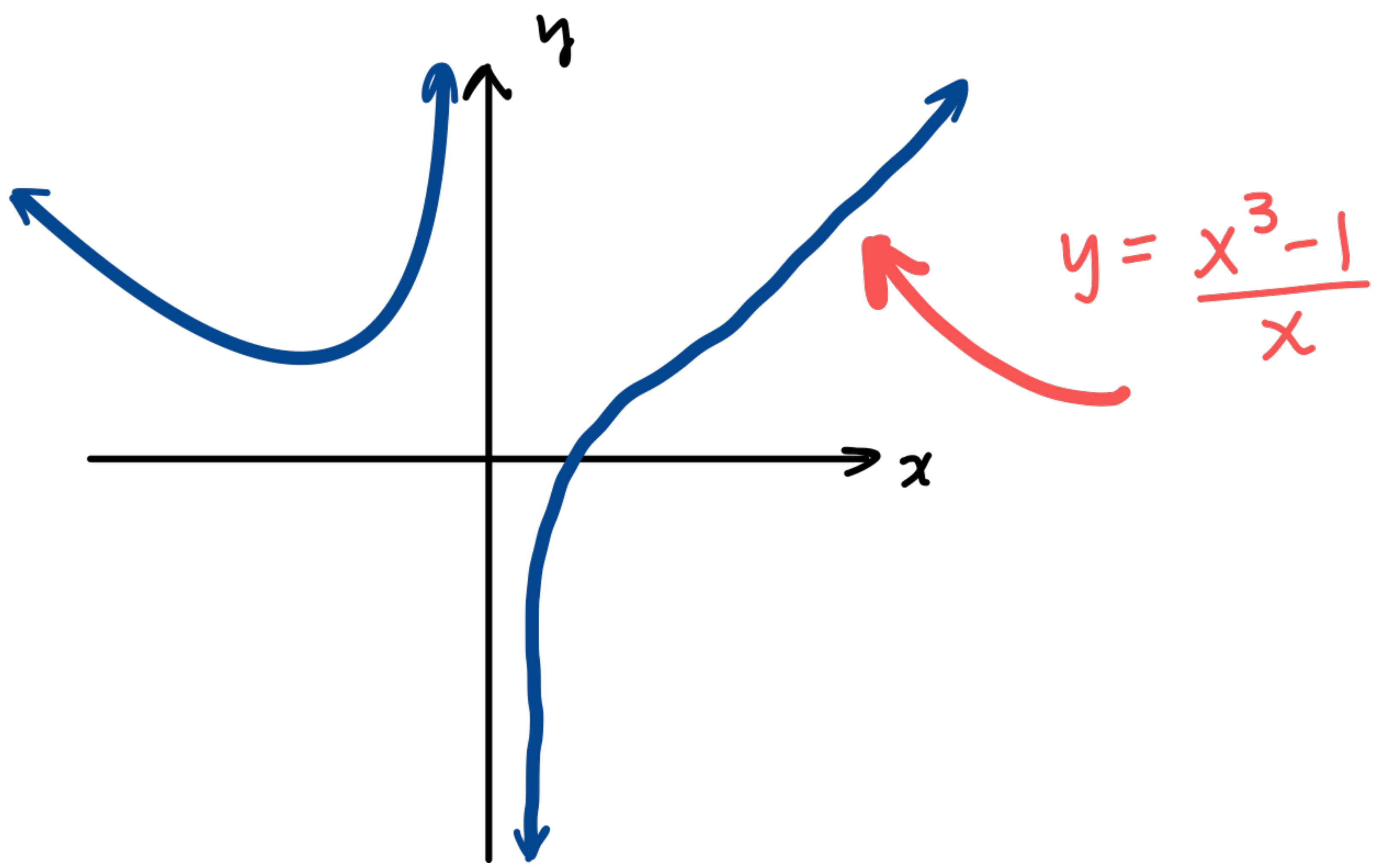
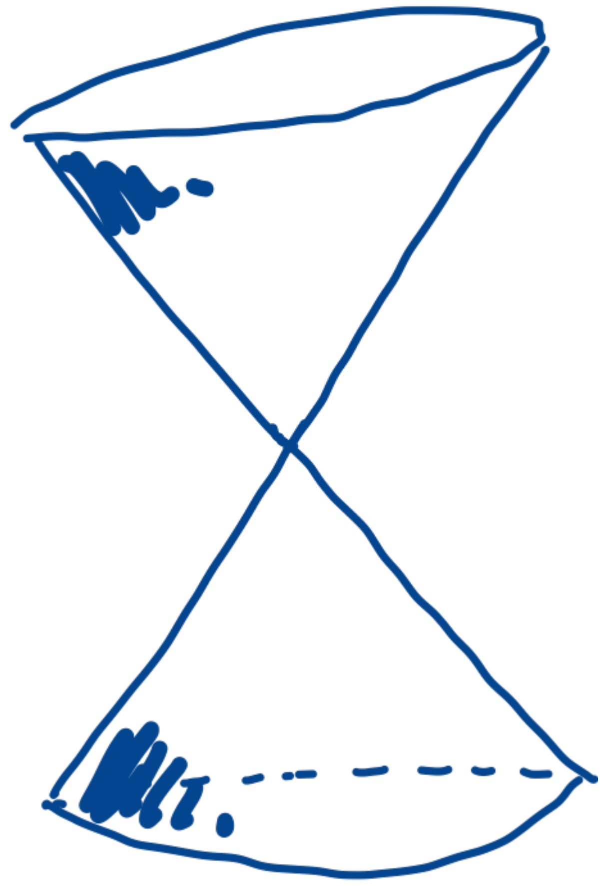
We'll call $V(f_1, \ldots, f_s)$ the *affine variety* defined by $f_1, \ldots, f_s$.

Ex $V(xy - x^3 + 1)$

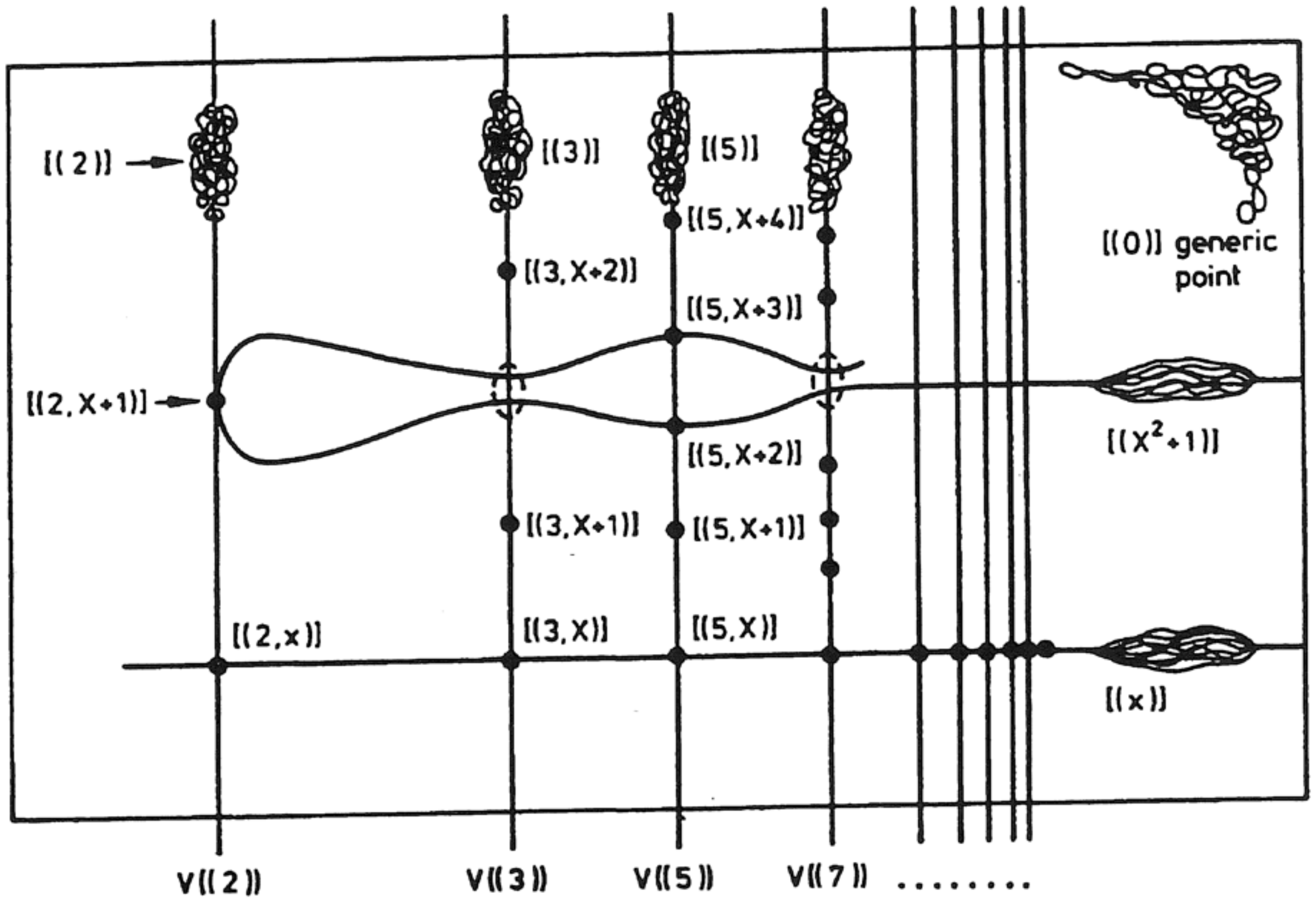


Ex. $V(z^2 - x^2 - y^2)$



Definition. Given a ring R, the *spectrum* of R written as $Spec\, R$ is the set of prime ideals of R.

Given an ideal I of a ring R, let V_I denote the set of prime ideals containing I. Define a topology (the *Zariski Topology*) on $Spec\, R$ by defining the collection of closed sets to be $\{V_I \mid I$ is an ideal of $R\}$.



A drawing of Spec $\mathbb{Z}[x]$ due to Mumford.

Pieter Belmans has compiled a collection of similar drawings - "***Atlas of Spec Z[x]***". See also "***Mumford's Treasure Map***".

Definition. An ideal $I \subset S = \mathbb{C}[x_1, \ldots, x_n]$ gives an *affine variety*

$$V(I) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } p \in V\}$$

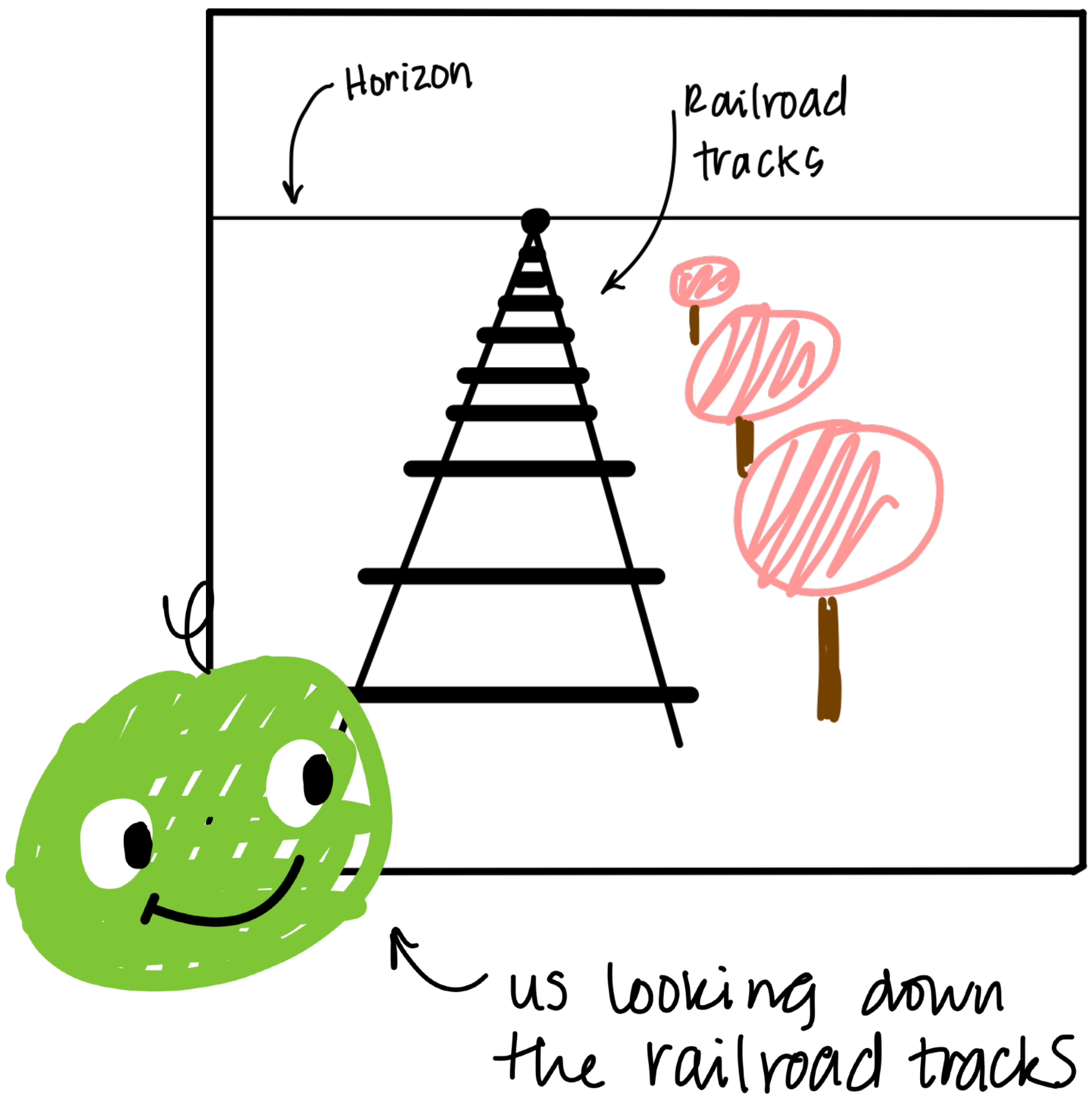
… on the other hand, an affine variety $V \subset \mathbb{C}^n$ gives us an ideal

$$I(V) = \{f \in S \mid f(p) = 0 \text{ for all } p \in V\}$$

Definition. To a variety V we can associate its *coordinate ring*

$$\mathbb{C}[V] = S/I(V)$$

… not all varieties are *affine.*


Horizon
Railroad tracks
us looking down the railroad tracks

Definition. $\mathbb{P}^2_{\mathbb{C}}$, the *complex projective plane*, is the collection of triples $(x_0, x_1, x_2) \in \mathbb{C}^3$ (with not all coordinates zero) endowed with the following equivalence relation:

$(x_0, x_1, x_2) \sim (y_0, y_1, y_2)$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $(x_0, x_1, x_2) = (\lambda y_0, \lambda y_1, \lambda y_2)$.

Ex. Consider the equation

$x^2 + y + z$. Does this define something in $\mathbb{P}^2$?

NO: Doesnt respect the equivalence relation

In $\mathbb{P}^2$ we should have $(1:1:1) \sim (3:3:3) \sim 3\cdot(1:1:1)$

but: if I plug in $(1,1,1)$ I get 3, and when I plug in $(3,3,3)$ I get $9+3+3=15$. but.. $15 \neq 3\cdot(1,1,1) = 3\cdot 3$

Ex. What do lines in $\mathbb{P}^2$ look like?

Consider: $y = x$ & $y = x+1$

to put these in $\mathbb{P}^2$, we'll homogenize.

$y = x \longrightarrow$ already homogeneous

$y = x+1 \longrightarrow y = x + z$

Notice: in $\mathbb{P}^2$, these lines intersect at the point $[1:1:0]$!

in $\mathbb{P}^2$, lines that were parallel intersect @ "point at infty" (hence railroad picture)

Silverman & Tate's ***Rational Points on Elliptic Curves*** contains an appendix on projective geometry.

in general, for $n \geqslant 1$ we can define projective n-space:

$$\mathbb{P}^n = \frac{\{[a_0, a_1, \dots, a_n] \mid a_0, a_1, \dots, a_n \text{ not all zero}\}}{\sim}$$

toric variety

Definition. The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. An *algebraic torus* T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism.

note: the n-dimensional torus is:
$(\mathbb{C}^*)^n = \mathbb{C}^n \setminus V(x_1 \cdots x_n)$

Definition. A *toric variety* is an (irreducible) variety V such that

1. $(\mathbb{C}^*)^n$ is a Zariski open subset of V, and
2. The action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on V.

Claim: $\mathbb{P}^2$ is a toric variety.

- is torus Zariski open subset?

yes: $(\mathbb{C}^*)^2 \xrightarrow{\sim} \mathbb{P}^2 \setminus V(x_0 x_1 x_2)$

- does torus action extend to an action on $\mathbb{P}^2$?

yes: $(t_1, t_2) \cdot (a_0 : a_1 : a_2) = (a_0 : t_1 a_1 : t_2 a_2)$

A common way to study projective varieties is to break them up into <u>affine patches</u> and study the affine pieces instead.

We'll do this for $\mathbb{P}^2$, and then use this construction to view $\mathbb{P}^2$ as a toric variety.

We'll let U_0 be the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ with $x_0 \neq 0$.

So: $U_0 = \{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_0 \neq 0 \}$

Since $x_0 \neq 0$, we can divide by x_0.

$$U_0 = \left\{ \left(\frac{x_0}{x_0} : \frac{x_1}{x_0} : \frac{x_2}{x_0} \right) \middle| \, x_0 \neq 0 \right\} = \left\{ \left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} \right) \right\}$$

So, U_0 is an affine variety. In fact - $U_0 \cong \operatorname{Spec} \mathbb{C}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \cong \mathbb{C}^2$.

* we can see this isomorphism via the map

$$(a_0, a_1, a_2) \longmapsto \left(\frac{a_0}{a_0}, \frac{a_1}{a_0}, \frac{a_2}{a_0} \right)$$

We'll do this same process with homogeneous coordinates x_1 and x_2 to get affine sets U_1 and U_2 respectively!

U_0

let U_0 be the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ with $x_0 \neq 0$.

So: $U_0 = \{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_0 \neq 0 \}$

$\cong \operatorname{Spec} \mathbb{C}[\frac{x_1}{x_0}, \frac{x_2}{x_0}]$

Since $x_0 \neq 0$, we can divide by x_0.

$U_0 = \{ (\frac{x_0}{x_0} : \frac{x_1}{x_0} : \frac{x_2}{x_0}) \mid x_0 \neq 0 \} = \{ (1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}) \}$

$\cong \mathbb{C}^2$

U_1

let U_1 be the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ with $x_1 \neq 0$.

So: $U_1 = \{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_1 \neq 0 \}$

$\cong \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}, \frac{x_2}{x_1}]$

Since $x_1 \neq 0$ we can divide by x_1.

$U = \{ (\frac{x_0}{x_1} : \frac{x_1}{x_1} : \frac{x_2}{x_1}) \mid x_1 \neq 0 \} = \{ (\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}) \}$

$\cong \mathbb{C}^2$

U_2

let U_2 be the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$ with $x_2 \neq 0$.

So: $U_2 = \{ (x_0 : x_1 : x_2) \in \mathbb{P}^2 \mid x_2 \neq 0 \}$

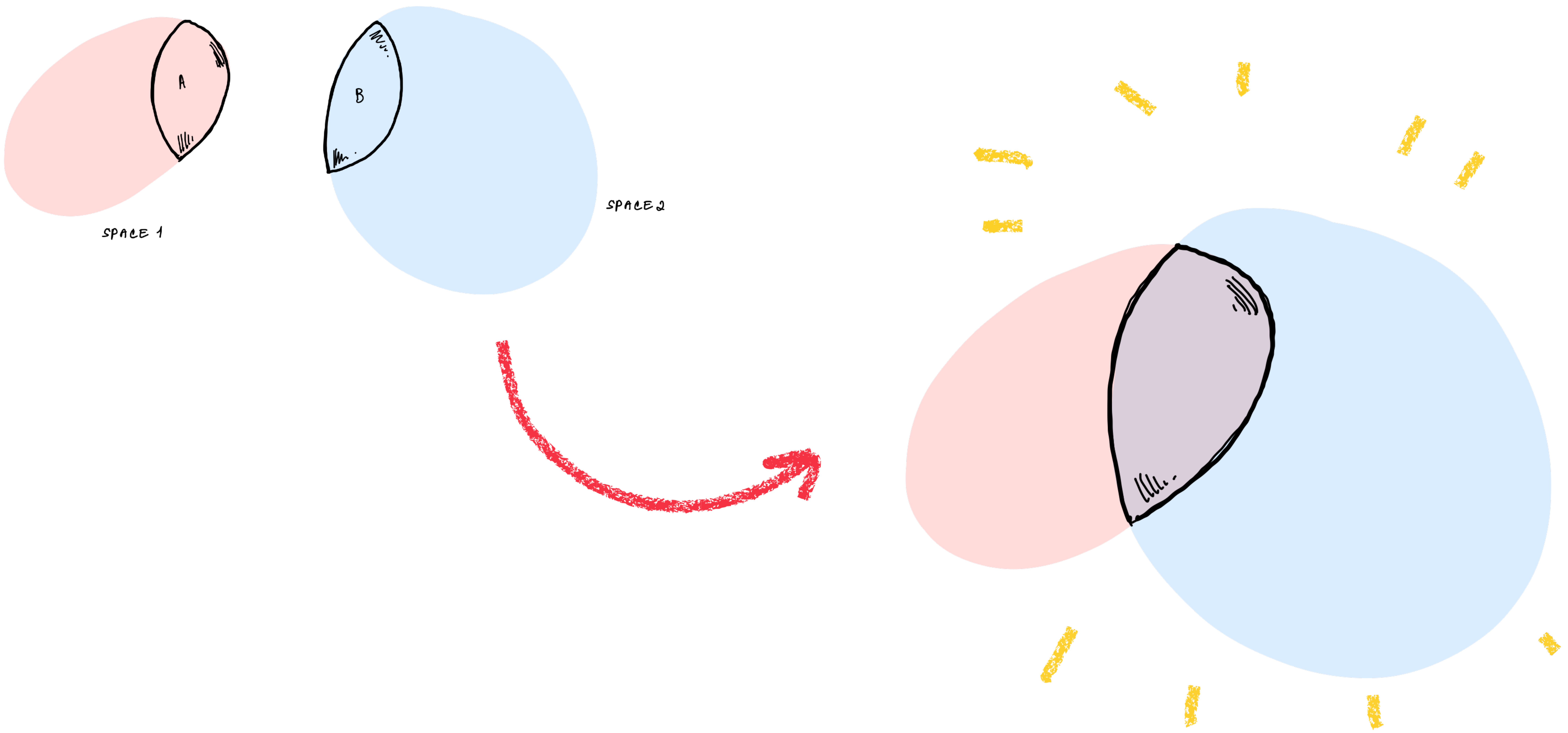
$\cong \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_2}, \frac{x_1}{x_2}]$

Since $x_2 \neq 0$, we can divide by x_2.

$U = \{ (\frac{x_0}{x_2} : \frac{x_1}{x_2} : \frac{x_2}{x_2}) \mid x_2 \neq 0 \} = \{ (\frac{x_0}{x_2} : \frac{x_1}{x_2} : 1) \}$

$\cong \mathbb{C}^2$

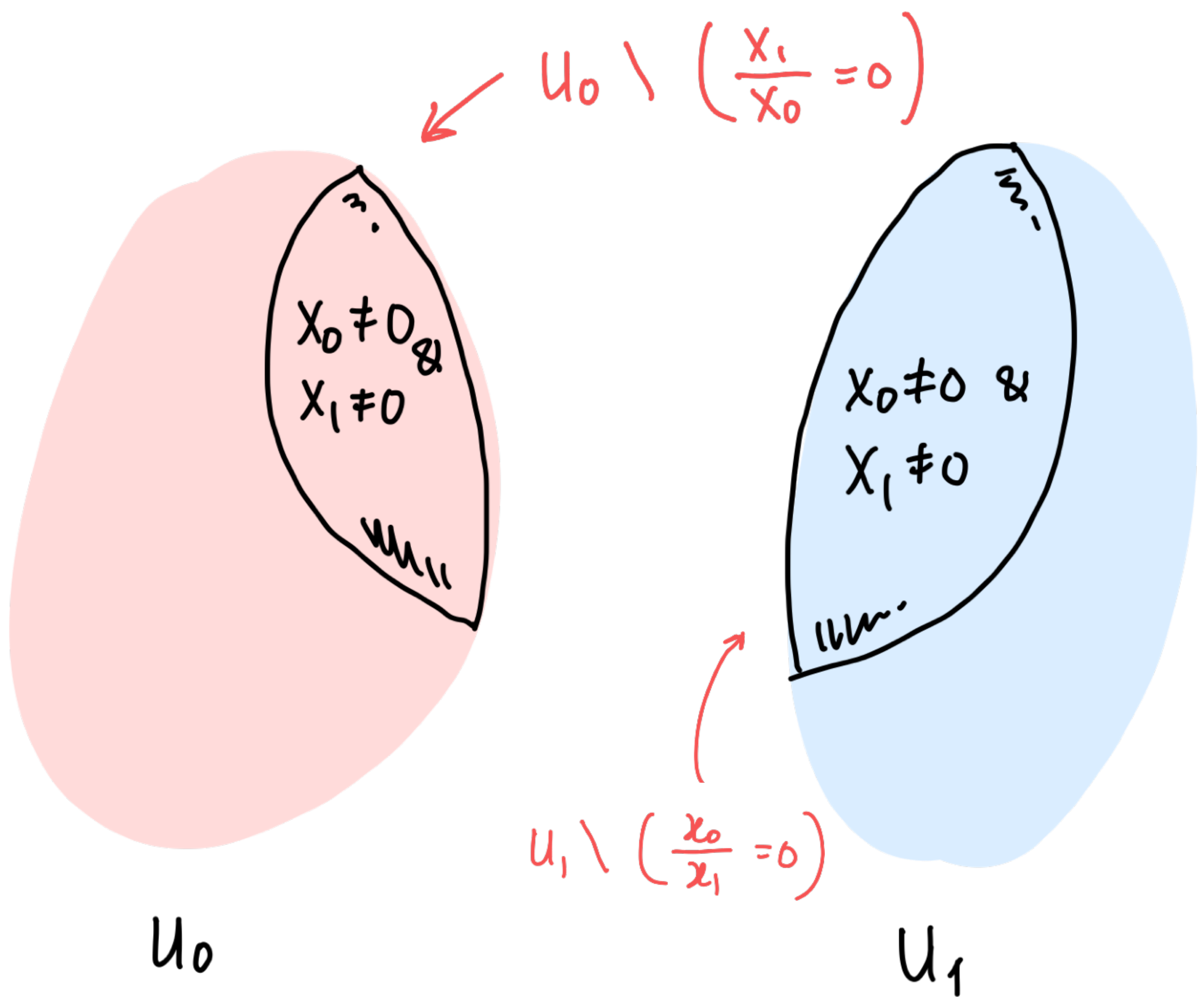
Now, we can "glue" U_0, U_1, and U_2 together to get $\mathbb{P}^2$



The gluing goes like this:

First: let's think about where U_0 & U_1 overlap.

This should happen when both x_0 and x_1 are nonzero.



We'll define a map

$$\left(U_0 \setminus \left(\frac{x_1}{x_0}=0\right)\right) \longrightarrow \left(U_1 \setminus \left(\frac{x_0}{x_1}=0\right)\right)$$

given by

$$\left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \longmapsto \left(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1}\right)$$

(show this is an isomorphism!)

+ we'll do the same thing for U_0 & U_2, U_1 & U_2, and U_0 & U_1 & U_2.

To save ourselves some time from now on, we'll do a change of coordinates.

Set $\frac{x_1}{x_0} =: x$ and $\frac{x_2}{x_0} =: y$.

Since $U_0 \cong \operatorname{Spec} \mathbb{C}[\frac{x_1}{x_0}, \frac{x_2}{x_0}]$ $\longrightarrow$ coordinate ring $\mathbb{C}[U_0] = \mathbb{C}[x, y]$

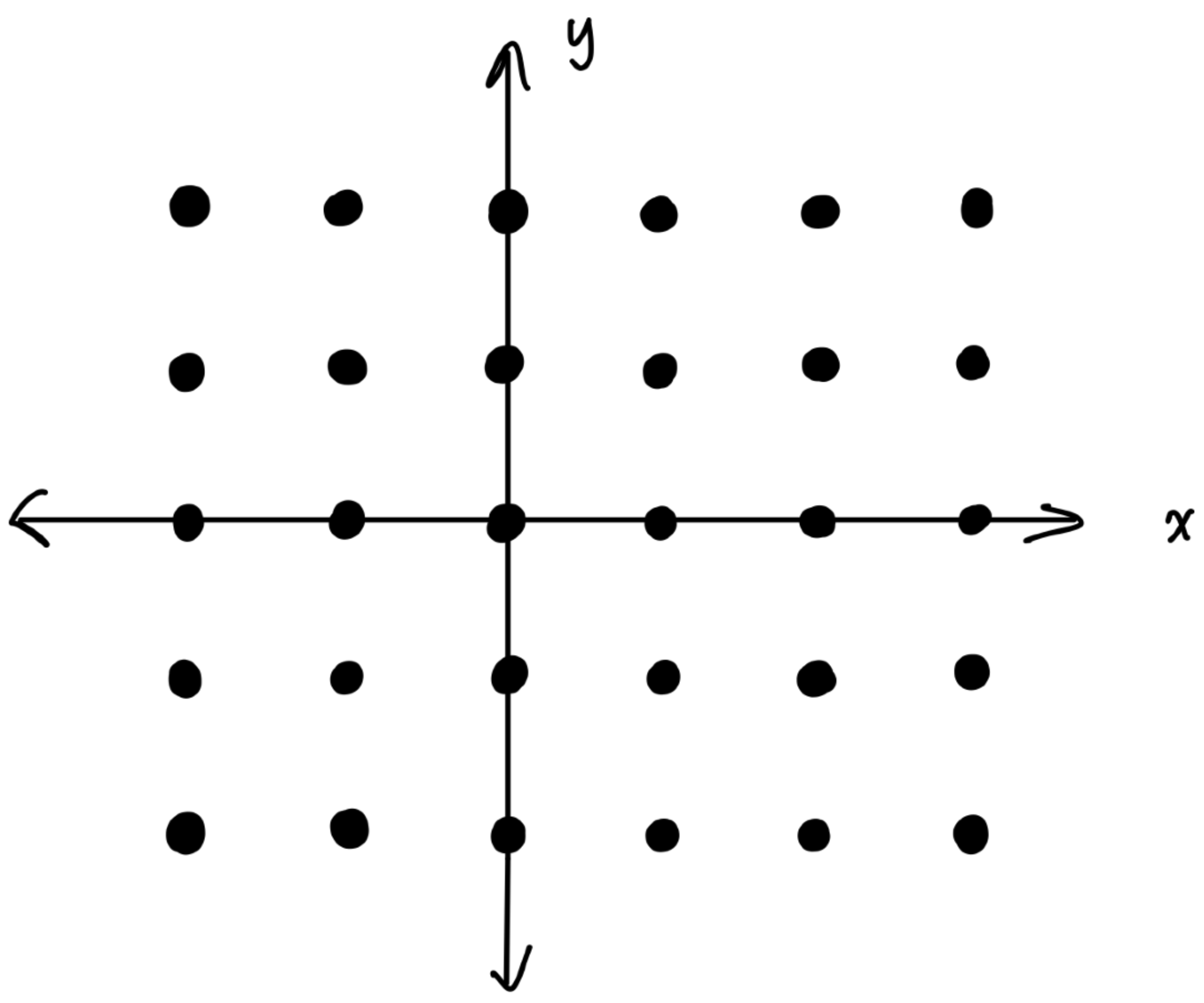
$U_1 \cong \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_1}, \frac{x_2}{x_1}]$ $\longrightarrow$ coordinate ring $\mathbb{C}[U_1] = \mathbb{C}[x^{-1}, x^{-1}y]$

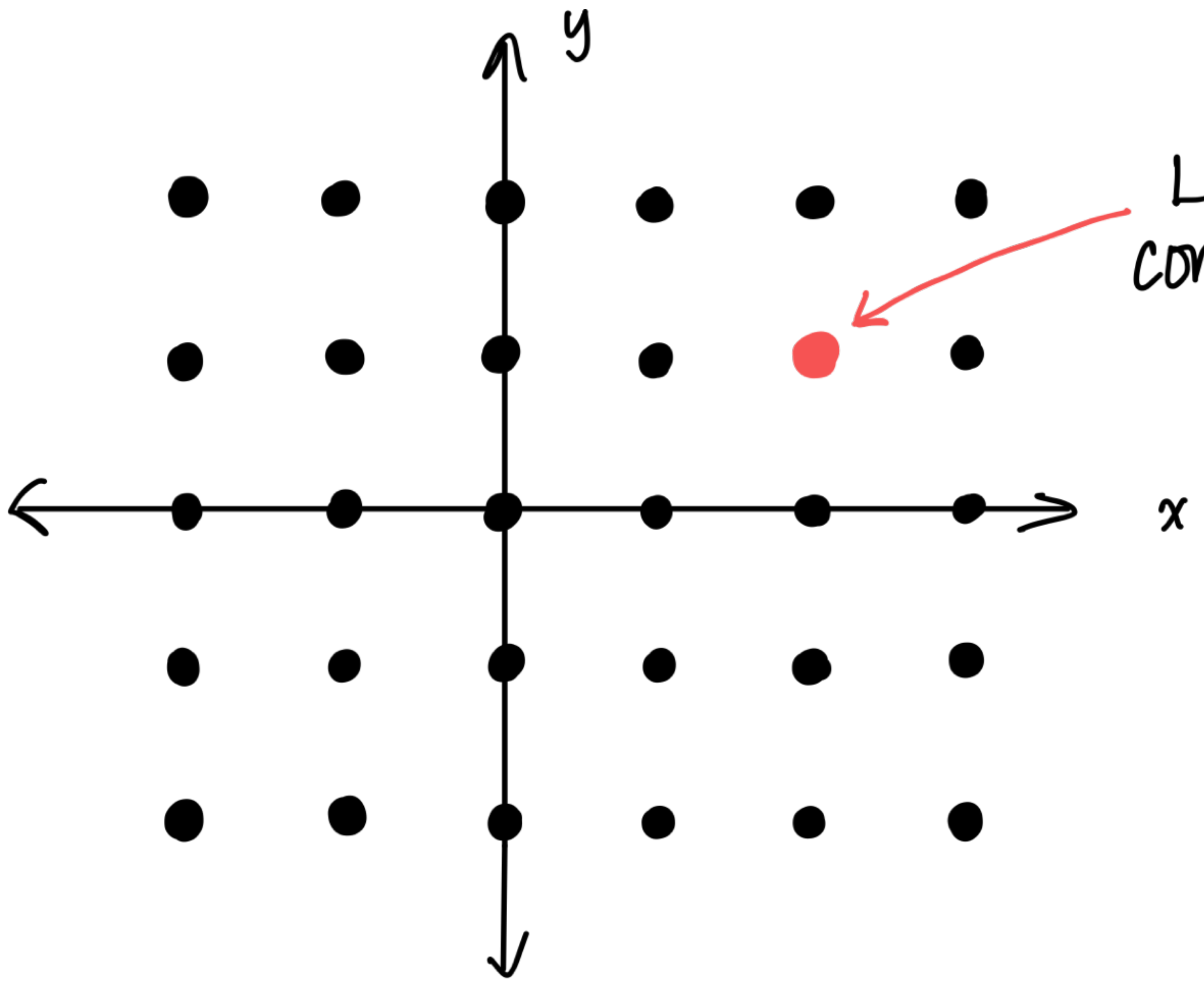
$U_2 \cong \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_2}, \frac{x_1}{x_2}]$ $\longrightarrow$ coordinate ring $\mathbb{C}[U_2] = \mathbb{C}[y^{-1}, xy^{-1}]$

How to view $\mathbb{P}^2$ as a toric variety?

We'll now view the monomials in $\mathbb{C}[U_i]$ as lattice points

Let $M \cong \mathbb{Z}^2$ be the lattice of Laurent monomials in variables x & y



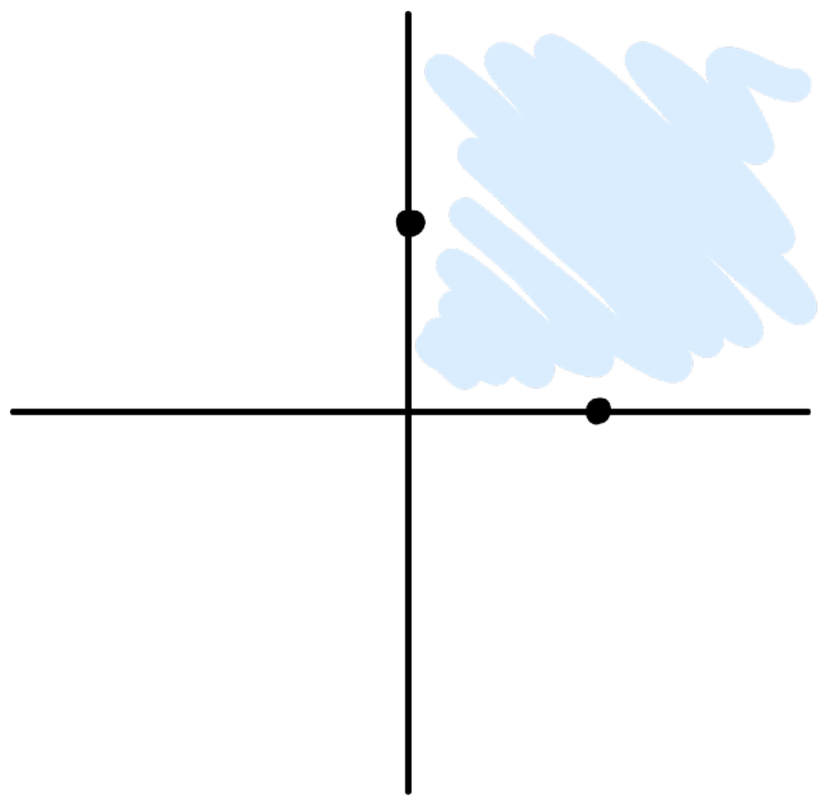


Lattice point (3,1) corresponds to the monomial x^3y^1

$(m,n) \in \mathbb{Z}^2$ will correspond to monomial $x^n y^m$.

What kind of monomials will we get from $\mathbb{C}[U_0] \simeq \mathbb{C}[x,y]$?

$$\begin{array}{cccccc} x, & y, & xy, & x^2y, & xy^2, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ (1,0) & (0,1) & (1,1) & (2,1) & (1,2) & \dots \end{array}$$



$\operatorname{span}_{\geq 0}\{(1,0),(0,1)\}$

$=$ monomials we get in $\mathbb{C}[U_0]$.

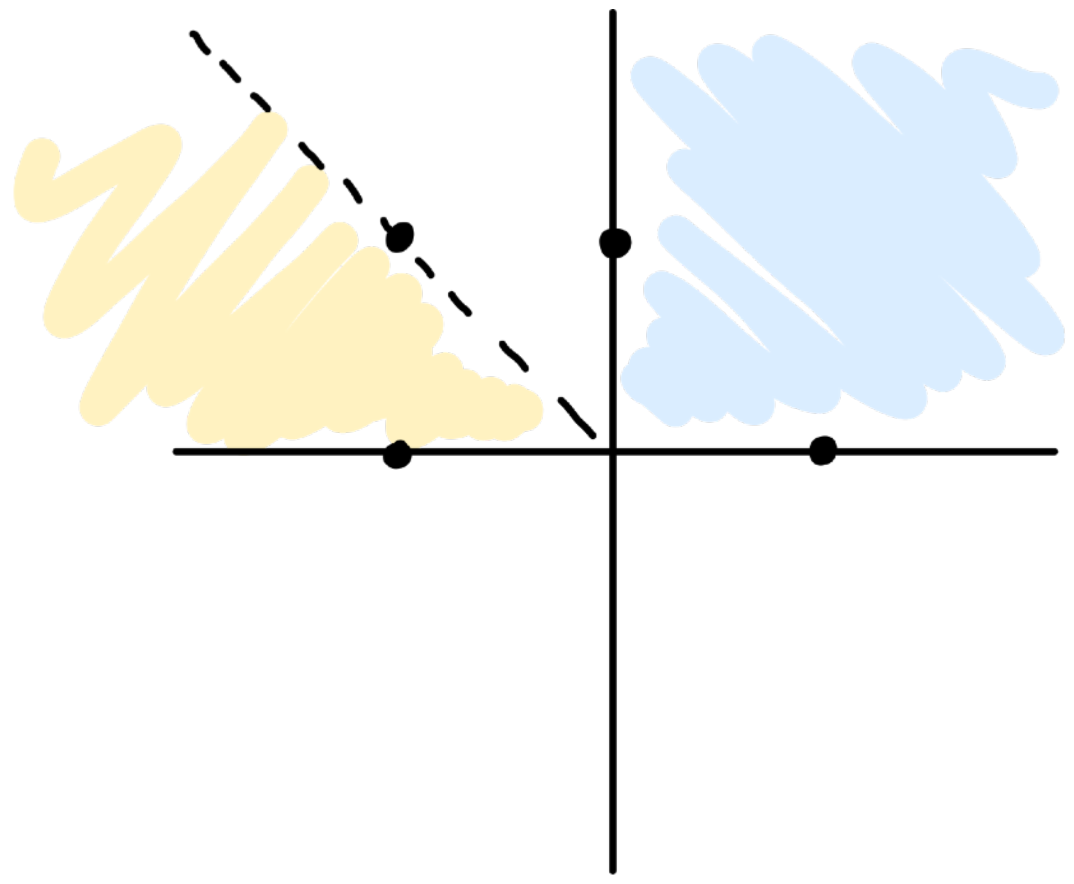
You can check the following

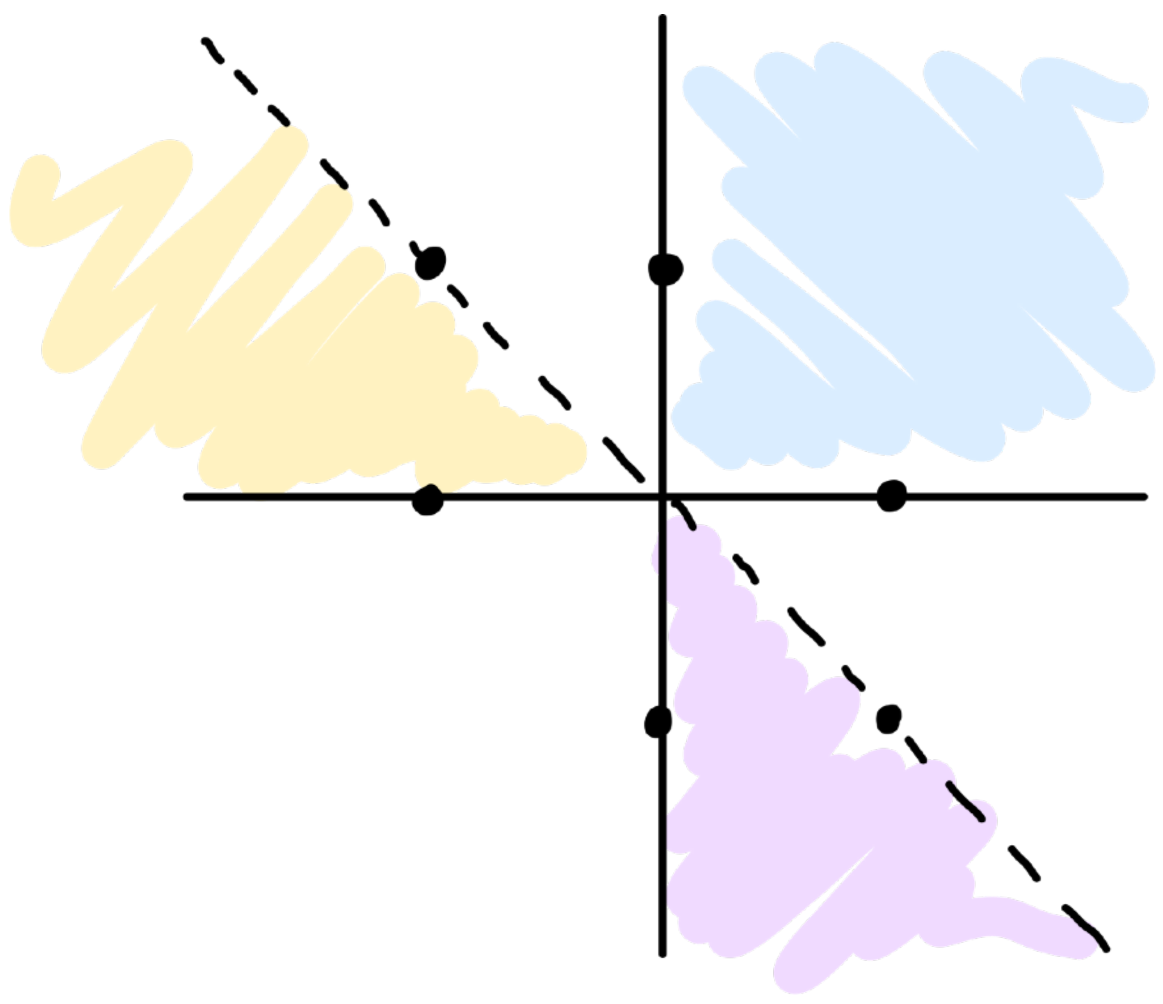
$\mathbb{C}[U_1]$ gives monomials

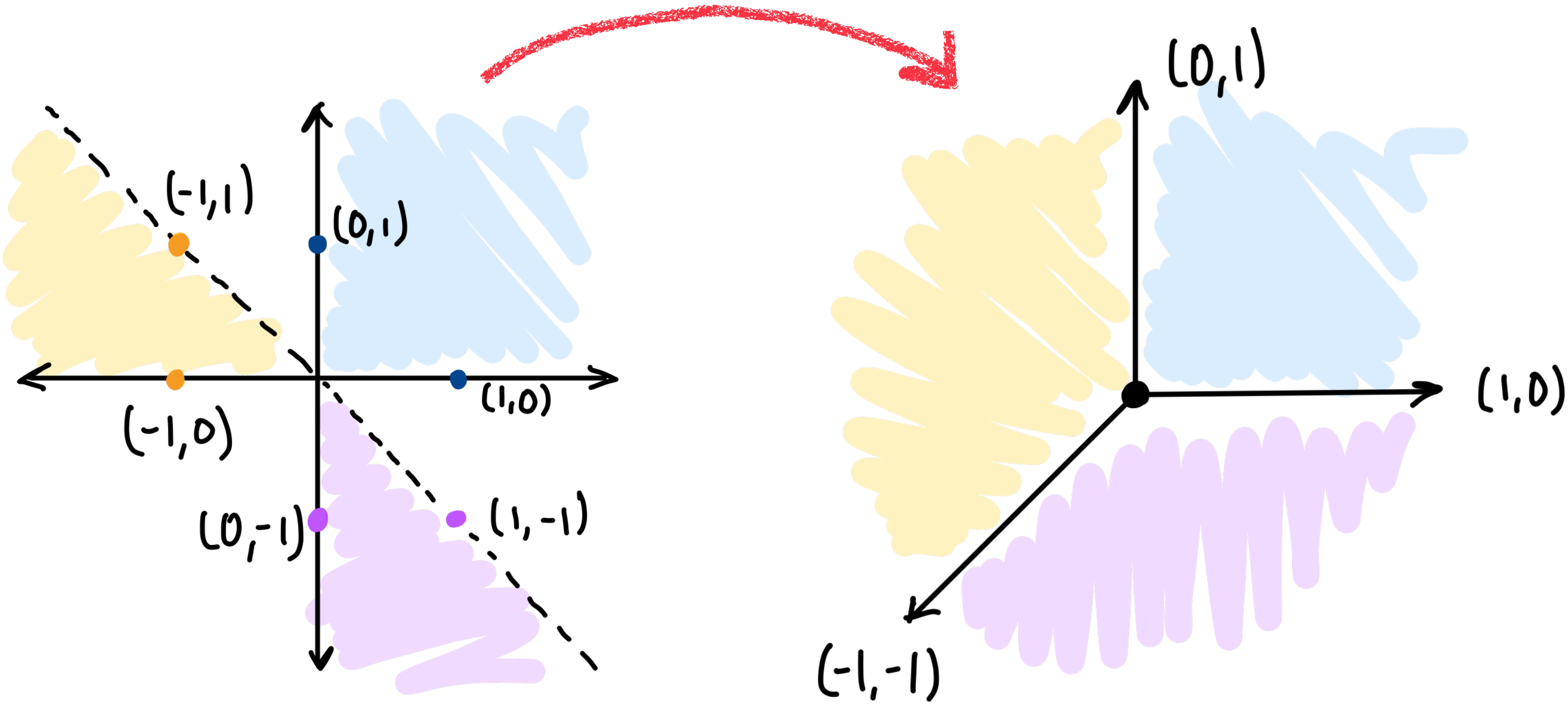
$\operatorname{span}_{\mathbb{Z}_{\geq 0}}\{(-1,0),(-1,1)\}$

$\mathbb{C}[U_2]$ gives monomials

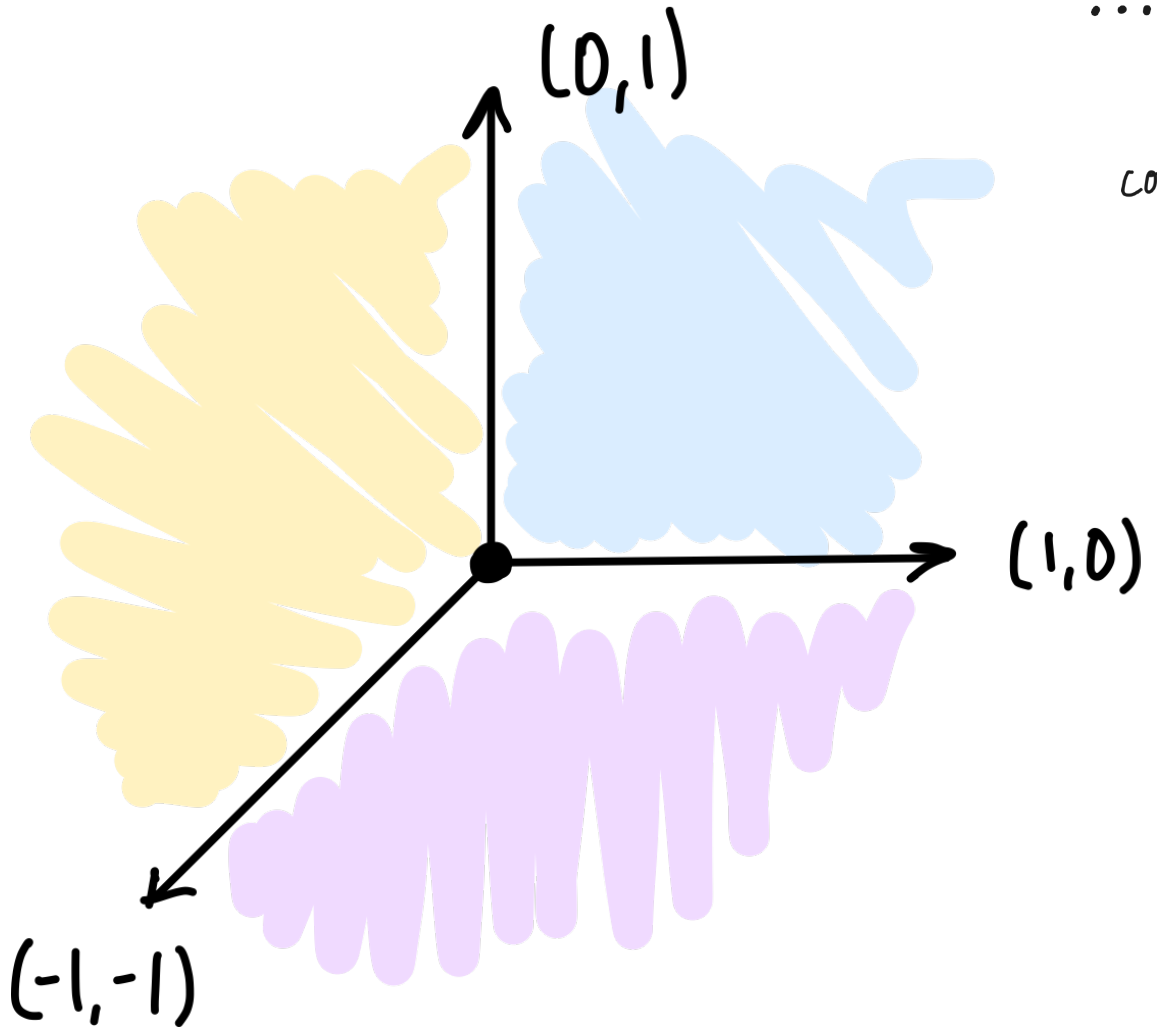
$\operatorname{span}_{\mathbb{Z}_{\geq 0}}\{(0,-1),(1,-1)\}$







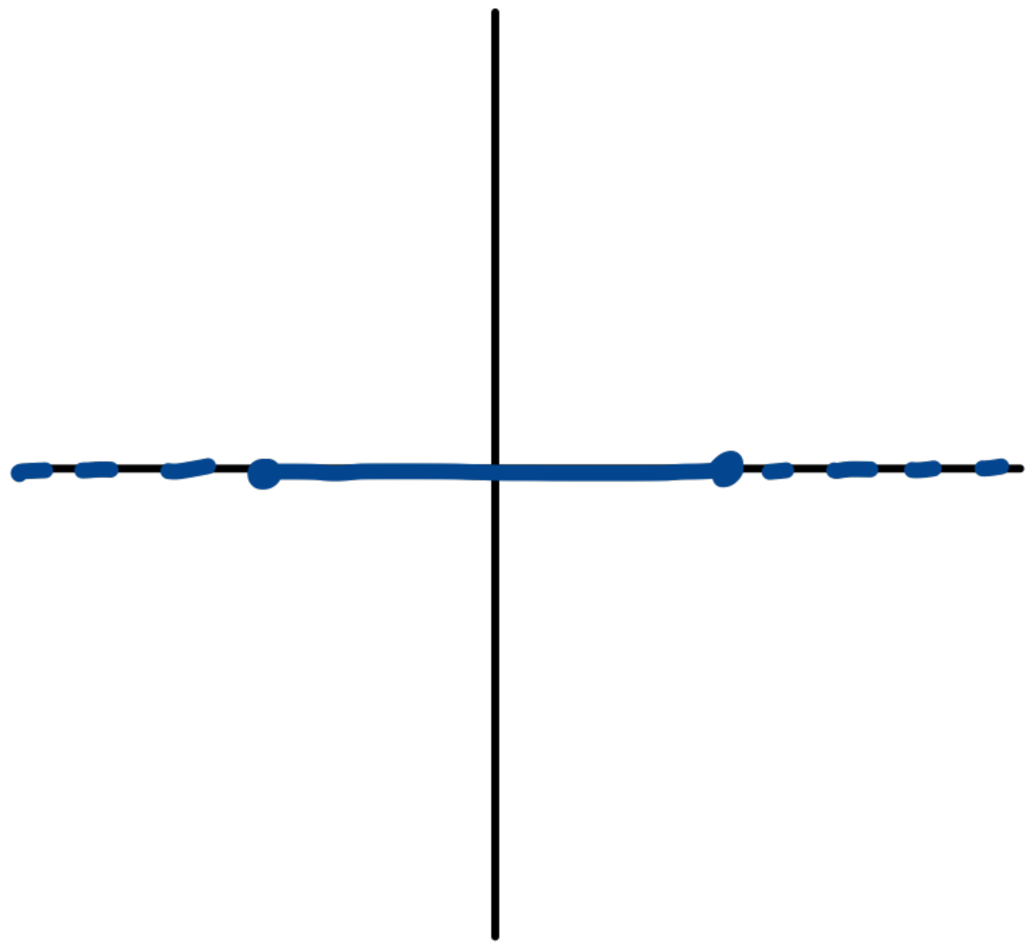
(-1,1)
(0,1)
(-1,0)
(1,0)
(0,-1)
(1,-1)
(0,1)
(1,0)
(-1,-1)



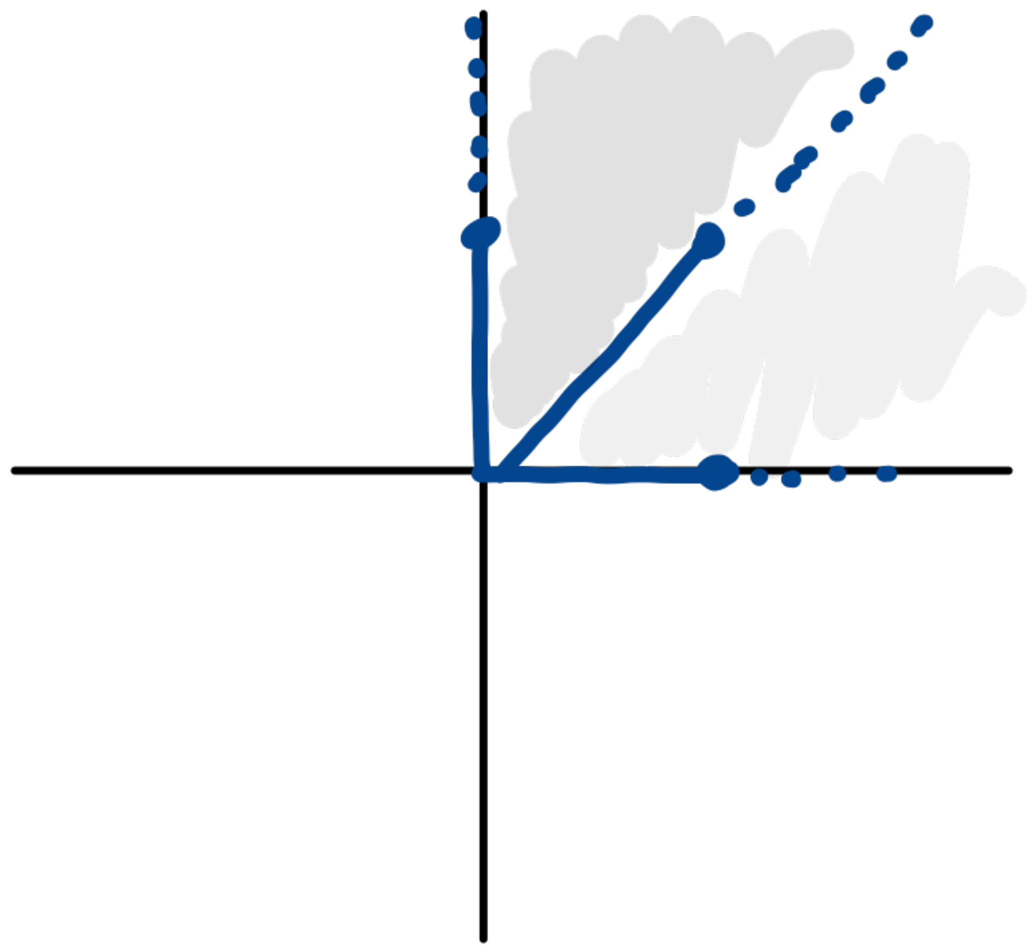
fan for $\mathbb{P}^2$

... why is this nice?

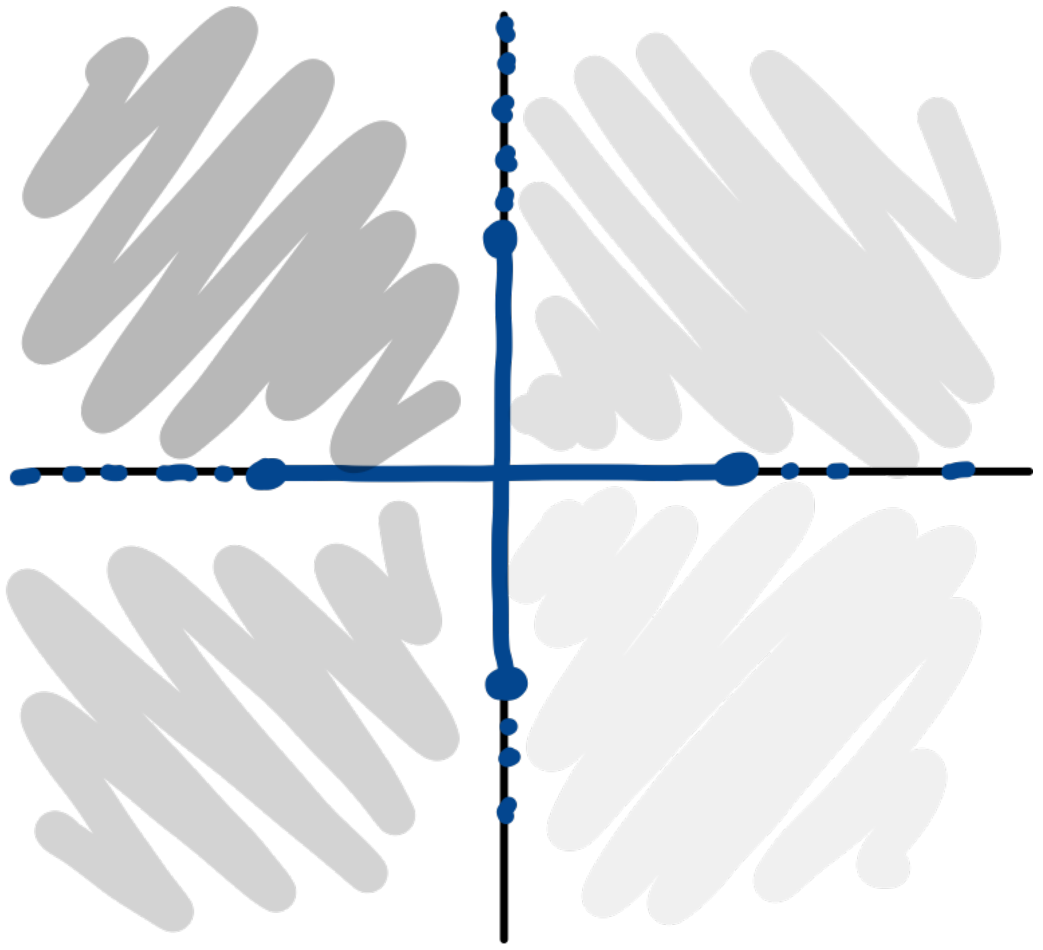
cones in fan $\xleftrightarrow{\text{bijection}}$ orbits of torus acting on variety



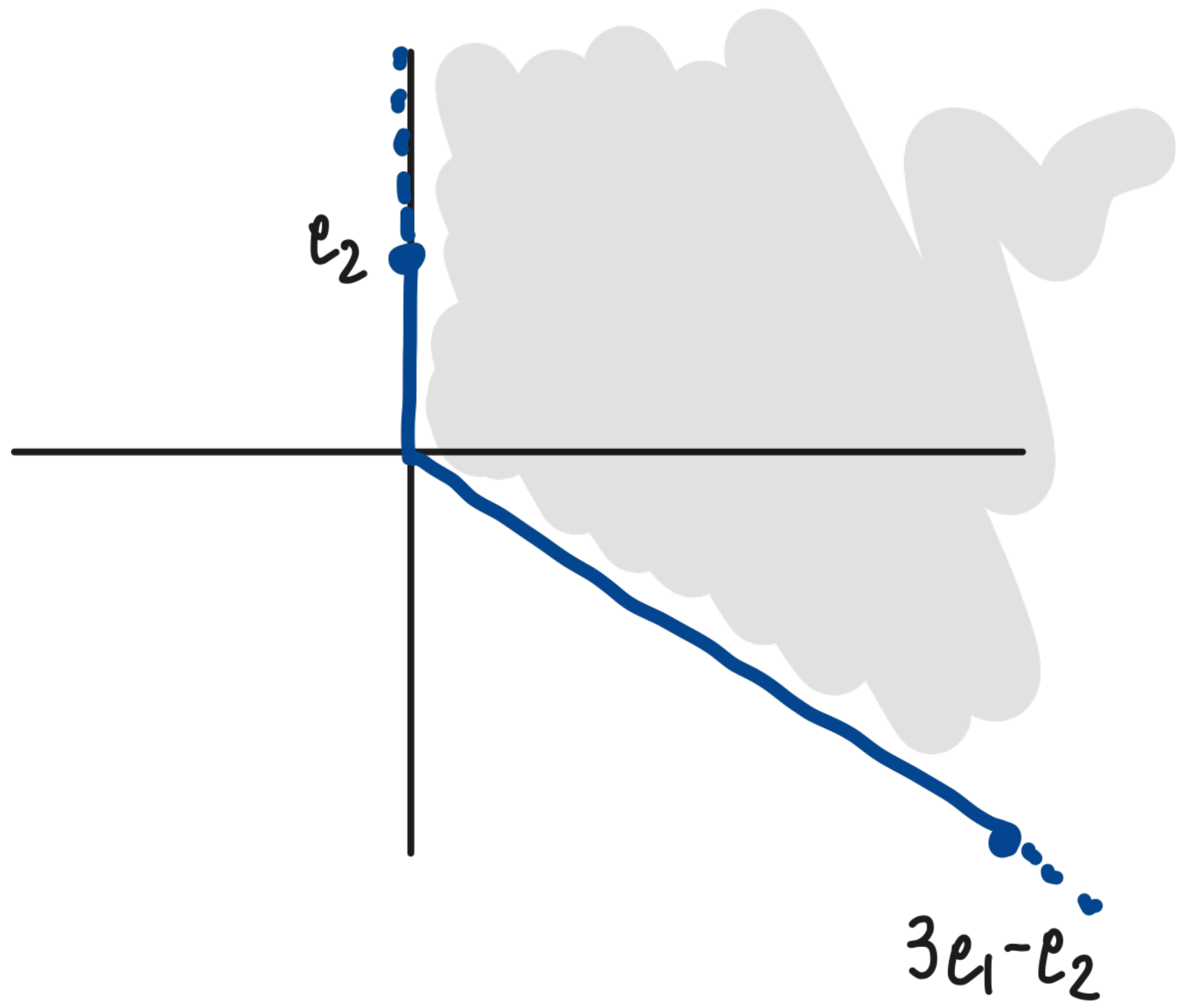
fan for $\mathbb{P}^1$



fan for $\mathrm{Bl}_0 \mathbb{C}^2$



fan for $\mathbb{P}^1 \times \mathbb{P}^1$



e_2
$3e_1 - e_2$

Thank you!