Examples of Spherical Varieties

references: • "Lectures on spherical and wonderful varieties" by Guido Pezzini
• "Intro to spherical varieties" by Boris Pasquier
• "Frobenius splitting Methods in Geometry + Rep Theory" by Michel Brion + Shrawan Kumar

Spherical varieties seminar – August 3rd 2018

GOAL: Discuss various examples of spherical, horospherical, and wonderful varieties, as well as any related definitions and theorems that we haven't seen in this seminar.

Notation/Defn Reminder

Def. • A G-variety is spherical if it is normal and has an open B-orbit.

• (X, x) is an embedding of G/H if X is spherical, the G-orbit Gx is open in X, and H is the stabilizer of $x \in X$. We call an embedding simple if it has a unique closed G-orbit

(It's also useful to recall that any spherical G-variety admits a cover by open G-stable simple spherical varieties – so we really only need to worry about simple guys.)

Def. • $\mathbb{C}(X)^{(B)} = \{ f \in \mathbb{C}(X) \mid bf = \chi(b)f \;\; \forall \;\; b \in B \;\&\; \text{some } \chi : B \to \mathbb{C}^* \}$ = "B-Eigenvectors" or B semi-invariant functions

• $\Lambda(X) = \{ \chi_f \mid f \in \mathbb{C}(X)^{(B)} \}$ = set of all B-weights

• A color is a B-stable prime divisor that is not G-stable. We call the set of all colors the palette, which we denote by $\Delta(X)$.

• Let (X, x) a simple embedding of G/H. Define $\mathcal{C}(X) \subseteq N(X)$ to be the convex cone generated by $\rho_x(\mathcal{D}(X))$ and by all of the G-invariant valuations associated to G-stable prime divisors of X. The pair $(\mathcal{C}(X), \mathcal{D}(X))$ is the painted cone of X.

• Given an embedding (X, x) of G/H, we define its painted fan as:

$$\mathcal{F}(X) = \{ \text{colored cones associated to } X_{Y,G} \text{ for any } G\text{-orbit } Y \text{ of } X \}$$

with $X_{Y,G} = \{ x \in G \mid \overline{G \cdot x} \supseteq Y \}$

Notation. • ρ_D = valuation associated to the prime divisor D.

• $N(X) := \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$

• $\mathcal{D}(X, Y)$ = colors of X that contain the closed orbit Y.

• $\mathcal{C}(X)$ = cone (convex) generated by $\rho_x(\mathcal{D}(X))$ & weights associated to G-stable prime divisors of X.

• $\mathcal{V}(X)$ = set of G-invariant valuations on X.

Pezzini's notes (and Rob's!) mention the following:

To classify ALL spherical G-varieties, we can look at the following:

- Fix a spherical subgroup $H \subseteq G$ and study all embeddings X of G/H
- Study all spherical subgroups $H \subseteq G$

(Recall: When G/H is a spherical variety, we call H a spherical subgroup.)

Revisiting examples of Spherical varieties → AND THEIR PAINTED FANS/CONES

First, I want to fully flesh out an example that we've already seen.

Example When Tracy talked about spherical embeddings, she gave the example of G/H with $G = SL_2$ and $H = T$. (It's also in Rob's notes from last week) During her talk, we saw that the homogeneous space G/H admits only one nontrivial embedding: $X = \mathbb{P}^1 \times \mathbb{P}^1$. We'll (try) to construct the painted fan for X.

Recall our choice of max torus $T = H$ and Borel subset B:

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \ a \neq 0 \right\} \quad \& \quad B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \ a \neq 0 \right\}$$

The B-orbit:

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ a^{-1} \end{pmatrix} \qquad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ a^{-1} \end{pmatrix}$$

Performing a change of variables as in Tracy's talk, we see that the B-orbit is iso. to

$$\left\{ (p,q) \in \mathbb{P}^1 \times \mathbb{P}^1 \text{ with } p \neq q \text{ and } p,q \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

(you can also check that this is open)

B-stable divisors: (that aren't G-stable)

From Tracy's talk, we have: $D^+ = \mathbb{P}^1 \times \{[1,0]\}$ & $D^- = \{[1,0]\} \times \mathbb{P}^1$

Notice also that the closed G-orbit is $Z = \mathrm{diag}(\mathbb{P}^1)$. We also have the following B-stable affine open set:

$$X_{Z,B} = X \setminus (D^+ \cup D^-) = \{[x,1],[y,1]\} \cong \mathbb{A}^2$$

From Rob's talk, recall that the function field of $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta(\mathbb{P}^1)$ is the same as that of $\mathbb{A}^2 - \mathbb{C}(x,y)$

In $X_{Z,B}$, we know a local equation for Z: $f(x,y) = (x-y)^{-1}$. This is a B-eigenvector of weight $-\alpha_1$.

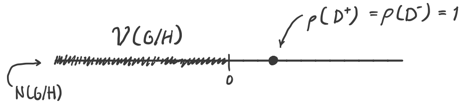
Thus, we have the following: $\langle \rho(Z), \alpha_1 \rangle = -1$, and $\mathcal{V}(X) = \mathbb{Q}_{\geq 0} v_Z$.

Notice also that $f(x,y)$ has poles of order 1 along D^+ and D^-, so that

$$\langle \rho(D^+), \alpha_1 \rangle = \langle \rho(D^-), \alpha_1 \rangle = 1.$$

Finally, from Rob's notes we have that $\Lambda(X) \cong \mathbb{Z}$, so that $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}) \cong \mathbb{Q}$.

Using this, we get the following painted cone:



Horospherical varieties

Def. A homogeneous space G/H is called <u>horospherical</u> if one of the following holds:

- It is a torus bundle over a flag variety G/P
- H contains the unipotent radical of a Borel subgroup
- H is the kernel of characters of a parabolic subgroup P of G

These are all equivalent

This is uniquely defined as the normalizer of H in G

Theorem X is horospherical if and only if $\mathcal{V}(X) = N(X)$.

Example Let $G = SL_2$ again. We'll take $H = U$ = set of unipotent upper Δ matrices in G.

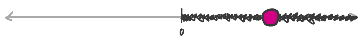
$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \; a \neq 0 \right\}$$

<u>NOTICE</u>: $G/H = SL_2/U = \mathbb{C}^2 \setminus \{(0,0)\}$.

- The only color D is given by the equation $\{y=0\}$. so y is a B-eigenvector with weight ω_1. $\Rightarrow \Lambda(SL_2/U) \cong \mathbb{Z}\omega_1$
- $\langle \rho(D), \omega_1 \rangle = \nu_D(f_{\omega_1}) = \nu_D(y) = 1$
- Notice: $\mathcal{V}(SL_2/U) \cong N(SL_2/U)$.

We have the following nontrivial <u>simple</u> embeddings:

- $(\mathcal{C}_1, \mathcal{D}_1) = (\mathbb{Q}_{\leq 0}\, \rho(D), \emptyset)$. According to Pezzini, this gives the embedding $G/H \cup \{\text{line @ } \infty\}$ ... $(\mathbb{P}^2 \setminus \{0\})$

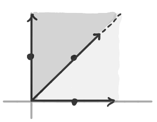


- $(\mathcal{C}_2, \mathcal{D}_2) = (\mathbb{Q}_{\geq 0}\, \rho(D), \{D\})$. This gives the embedding $G/H \cup \{(0,0)\} \cong \mathbb{C}^2$.

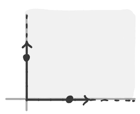
- $(\mathcal{C}_3, \mathcal{D}_3) = (\mathbb{Q}_{\geq 0}\, \rho(D), \emptyset)$. This gives us $Bl_0\, \mathbb{C}^2$.

This is possibly a stupid question, but how do these painted fans compare to their toric variety counterparts? (ie $Bl_0\, \mathbb{C}^2$... as a toric variety)

ie:



$Bl_0\, \mathbb{C}^2$



$\mathbb{C}^2$

(The shaded regions are only to indicate the maximal cones of the fan)

... What about the nonsimple embeddings of SL_2/U? There are two:

↳ painted fan instead of a painted cone!

- $\mathcal{F} = \{ (\{0\}, \emptyset), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_2, \mathcal{D}_2) \}$ gives embedding $\mathbb{P}^2$. The fan is the following:

- $\mathcal{F} = \{ (\{0\}, \emptyset), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_3, \mathcal{D}_3) \}$ which gives $Bl_0 \mathbb{P}^2$

Wonderful Varieties

Def. Let X be a G-variety. We call X <u>wonderful</u> if:

(i) X is smooth and complete

(ii) X contains an open G-orbit X_G° whose complement is the union of smooth G-stable prime divisors $X^{(1)}, \dots, X^{(r)}$ which have normal crossings and nonempty intersection.

(iii) For all $x, y \in X$ we have:

$$Gx = Gy \text{ if and only if } \{ i \mid x \in X^{(i)} \} = \{ j \mid y \in X^{(j)} \}$$

The number r (of divisors from part (ii)) is the <u>rank</u> of X, and $\bigcup_{i=1}^{r} X^{(i)}$ (union of G-stable prime divisors) is the <u>boundary</u> of X, which we denote as ∂X.

[A note on "normal crossing": I interpret this to mean that the $X^{(i)}$ intersect like the coordinate hyperplanes in $\mathbb{C}^n$. As in Pezzini's notes, any intersection of them will give a wonderful subvariety.]

Theorem (due to Luna in 1996) Wonderful varieties are projective and spherical.

Def Let $z \in X$ be the unique point fixed by B_-. It lies on $Z = Gz$ = unique closed G-orbit. Consider the vector space $T_z X / T_z Z$. (this is naturally a T-module) Its T-weights are called spherical roots of X, and we denote the set of these by Σ_X.

If X is the wonderful embedding of a S.H.S. G/H of rank r, then the spherical roots are in bijection with codim 1 G-stable closed subvarieties, and with codim $r-1$ G-stable closed subvarieties.

Facts (That I found interesting)

- Flag varieties are wonderful of rank 0.
- The only wonderful variety that is toric and wonderful is the point.
- A spherical variety is wonderful if + only if its the canonical embedding of its open G orbit and the embedding is smooth.

  Thus: a spherical homogeneous space admits at most one wonderful embedding
- Wonderful varieties are classified by their associated root system
- A natural question: when does a spherical homog. space admit a wonderful embedding?

  $\rightsquigarrow$ Classification of such spaces is not yet complete.

  Although, there is a necessary condition: $N_G H / H$ must be finite.

The general situation: The homogeneous spaces $G/H = H \times H / \mathrm{diag}(H)$ for H semisimple and adjoint admit wonderful compactification X. X has spherical roots $\sigma_i = \alpha_i + \alpha_i'$, where α_i and α_i' are simple roots for each copy of H. There are also n colors, & we can determine the values of their functionals on each σ_i via the cartan matrix assoc. to H.

According to Pezzini...

Note. SL_2 admits a wonderful compactification. (Its the only non-adjoint simple group that does) it is:

$$X = \{ad - bc = t^2\} \subset \mathbb{P}(M_{2\times 2} \oplus \mathbb{C})$$

not so

$\boxed{\text{A wonderful compactification}}$ → to Alicia..

Example. Let $G = PGL_2$. We consider $G \times G$, with $X = \mathbb{P}(M_{2\times2}) \cong \mathbb{P}^3$

$\mathcal{B}$ = Borel subgroup of $G\times G$.. so we'll take $\mathcal{B} = B^- \times B$, with:

$B^- = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right\}$ and $B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$. Notice that $B^- \cap B = T$.

- What is the $\mathcal{B}$-orbit of X? (How does $\mathcal{B}$ act on the identity?)

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c^{-1} & 0 \\ -d & c \end{pmatrix} = \begin{pmatrix} ac^{-1} - bd & bc \\ -a^{-1}d & a^{-1}c \end{pmatrix}$$ → Need $a \neq 0$.

(Also.. didn't Keller mention this in his talk? ... except with $SL_2 \times SL_2$?)

so the open $\mathcal{B}$-orbit will be: $X_B^o = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \neq 0,\ ad-bc \neq 0 \right\}$

- What is the G-orbit of X?

This should just be: $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ ad-bc \neq 0 \right\}$ → Now we can figure out what our boundary divisors should be.

so, the open $G\times G$-orbit of X will be: $X^o = \mathbb{P}(M_{2\times2}) \setminus Z(ad-bc)$

then.. $X \setminus X^o$ ought to give us the boundary divisors. Notice $\mathbb{P}(M_{2\times2}) \setminus \big(\mathbb{P}(M_{2\times2}) \setminus Z(ad-bc)\big) = \underbrace{\boxed{Z(ad-bc)}}_{\text{boundary divisor}}$

- What about the colors?

Recall from the defn: $X^o \setminus X_B^o$ = union of prime B-stable divisors = union of colors

so.. $X^o \setminus X_B^o = (\mathbb{P}^3 \setminus Z(ad-bc)) \setminus (\mathbb{P}^3 \setminus (Z(ad-bc) \cup Z(a))) = \boxed{Z(a)}$ → one color!

(See "Total coord ring of a Wonderful variety" by Brion)

- Spherical roots (w.r.t $B^- \times B$) are pairs $(-\alpha_i, \alpha_i)$, where α_i are the simple roots of the root system associated to PGL_2... which is A_1. A choice of simple root for A_1 is $e_1 - e_2$.

So... $\Lambda(X) \cong$ root lattice of A_1, and $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q}) \cong$ weight lattice of A_1. (tensored w/ $\mathbb{Q}$?)

Finally... $\mathcal{V}(X)$ = negative Weyl chamber. (I think Blake talked about this?)

an attempt at a painted fan: