Derived Categories, Arithmetic, & Rationality

Alicia Lamarche

CMS Winter Meeting
December 8, 2020

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Slides can be downloaded from *alicia.lamarche.xyz/talks/12082020.pdf*

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forthcoming work with:

Matthew Ballard
University of South Carolina
matthewrobertballard.com

arXiv:????.?????

Also contains

Joint work with

Matthew Ballard
University of South Carolina
matthewrobertballard.com

Alexander Duncan
University of South Carolina
people.math.sc.edu/duncan

Patrick McFadden
Fordham University
mcfaddin.github.io

arXiv:2006.06876 & **arXiv:2009.10175**

Question:

Can $D^b(X)$ be used to answer rationality questions about X?

X will denote a smooth projective variety throughout.

outline

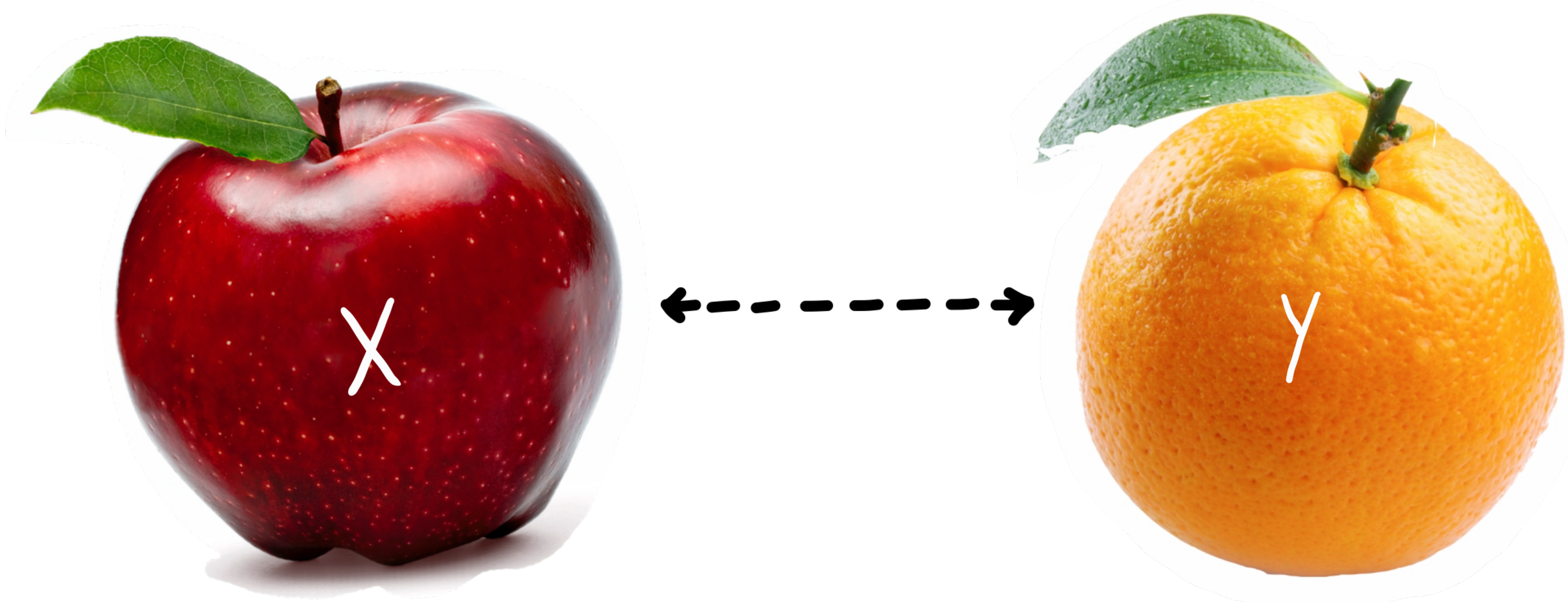
1. Motivation
2. Derived Categories
3. Arithmetic Toric Varieties
4. Results

Notation. For a smooth projective variety X, $D^b(X) := D^b(\text{Coh}(X))$

bounded derived category of coherent sheaves on X

We say that a variety X defined over a field k has a ***rational point*** (or k-***rational point***) if there is a map $\text{Spec}(k) \to X$.

Definition. A variety X defined over a field k is called **rational** if it is *birational* to $\mathbb{P}^n$ for some n.

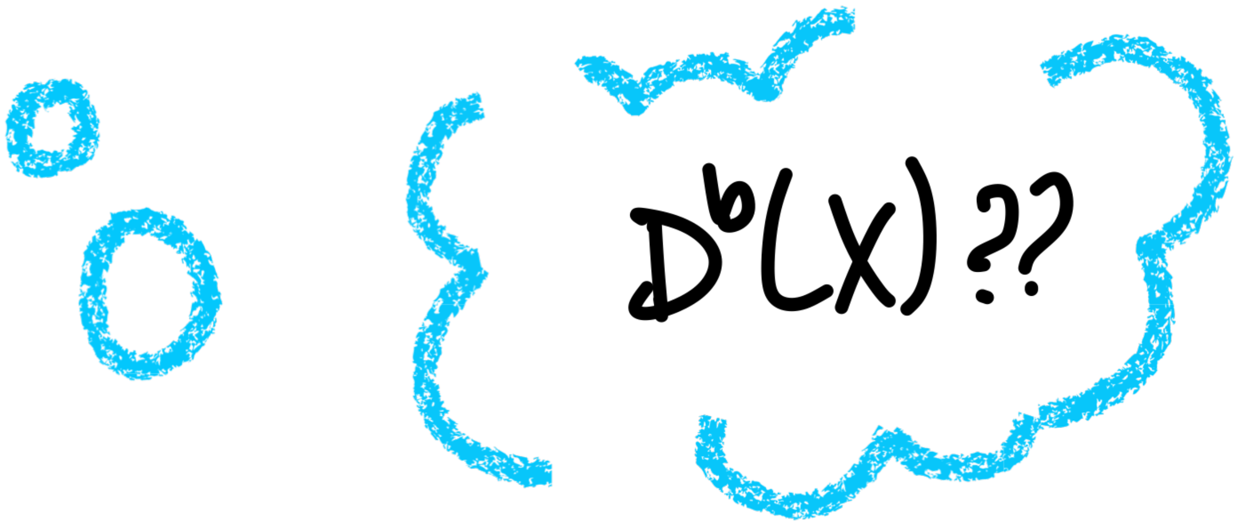


How can you determine if two varieties are birational?

Theorem. *(Weak Factorization)* Any birational map between two smooth complex varieties can be decomposed as a series of finitely-many blow-ups/blow-downs along smooth subvarieties.

Naive wishlist for a tool to detect rationality:

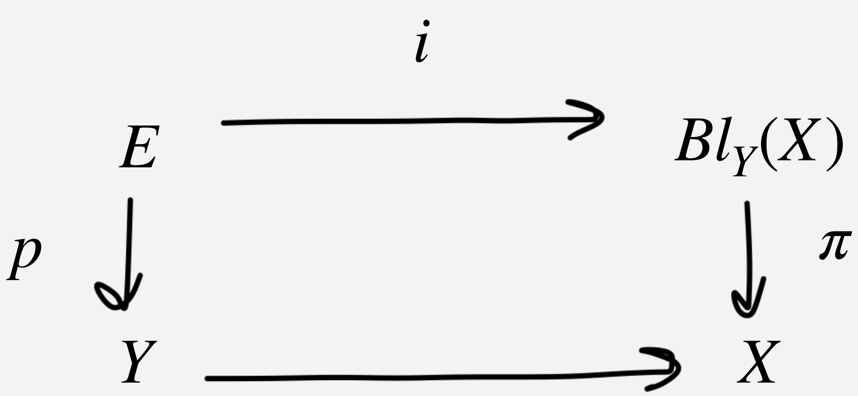
- Is relatively 'nice' for $\mathbb{P}^n$
- Behaves with respect to weak factorization



Theorem. (*Beilinson*) $D^b\left(\mathrm{Coh}(\mathbb{P}^n)\right)$ 'decomposes' in the following way:

$$D^b\left(\mathrm{Coh}(\mathbb{P}^n)\right) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle$$

Theorem. (*Bondal/Orlov*) Let $Y \subset X$ a smooth subvariety of codimension c.



Then, we have the following decomposition of $D^b(Bl_Y(X))$:

$$D^b(Bl_Y(X)) = \langle \pi^* D^b(X), i_* p^* D^b(Y), i_* p^* D^b(Y) \otimes \mathcal{O}_p(1), \ldots, i_* p^* D^b(Y) \otimes \mathcal{O}_p(c-2) \rangle$$

A vague idea:

$$X \text{ rational} \Longrightarrow D^b(X) \text{ is not `too' complicated?}$$

$$D^b(X) \text{ is not `too' complicated} \Longrightarrow X \text{ rational?}$$

How do we 'measure' how complicated $D^b(X)$ is?

Conjecture. (*Orlov)* A smooth projective variety with a full exceptional collection is rational.

Conjecture. (*Lunts)* Over a general field k, a full k-exceptional collection for $D^b(X)$ implies that X admits a locally-closed stratification into subvarieties that are each k-rational.

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Conjecture. (*Lunts)* Over a general field k, a full k-exceptional collection for $D^b(X)$ implies that X admits a locally-closed stratification into subvarieties that are each k-rational.

Theorem. (***Ballard, Duncan, L., McFaddin***) A smooth projective toric variety X over a field k with $X(k) \neq \emptyset$ possessing a full k-exceptional collection is rational.

Question. For X defined over a field k, if $D^b(X)$ admits a particular type of 'nice' decomposition, does this mean that X is rational over k?

Evidence:

For a Severi-Brauer curve X, $D^b(X)$ can be decomposed, but is only 'nice' when X is trivial.

Dimension 1:

- Genus 0: nontrivial Severi-Brauer curves never have k-exceptional collections. (Unless they are trivial- i.e. $\mathbb{P}^1_k$)
- Genus ≥ 1 curves do not admit nontrivial semiorthogonal decompositions.

(Okawa, 2011)

Question. For X defined over a field k, if $D^b(X)$ admits a particular type of 'nice' decomposition, does this mean that X is rational over k?

Evidence:

Dimension 2:

- **Vial (2016)**
  { full k-exceptional collection & geometrically rational } $\implies$ rational
- **Brown & Shipman (2015)**
  { full strong exceptional collection of line bundles & stability conditions } $\implies$ rational
  (over $\mathbb{C}$)

Question. Can $D^b(X)$ be used to answer rationality questions about X?

Question. If X is defined over a field k and $D^b(X)$ admits a particular type of 'nice' decomposition, does this mean that X is ~~rational~~ over k?

stably rational

retract rational

unirational

...

Question. For X defined over a field k, if $D^b(X)$ admits a particular type of 'nice' decomposition, does this mean that X has a ***rational point*** over k?

The derived category can be used to detect the difference between 'generalized Del Pezzo varieties' and their twists by torsors.

Theorem. (*Ballard, L.*) If X is a generalized Del Pezzo variety, then X has a rational point if and only if $D^b(X)$ admits a full étale exceptional collection.

Question. For X defined over a field k, can $D^b(X)$ detect the existence of k-rational points on X?

Addington, Antieau, Frei, Honigs (2019)
Gave an example of two derived equivalent varieties where one has a rational point and the other does not.

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<u>Definition.</u> Let A be a finite dimensional k-algebra of finite homological dimension. An object E of $D^b(X)$ is A***-exceptional*** if:

- $\mathrm{End}(E) \cong A$
- $\mathrm{Hom}(E, E[n]) = 0$ for all $n \neq 0$

An object E is ***exceptional*** if it is A-exceptional for a division algebra A, and is ***étale exceptional*** if A is a finite separable field extension of k.

A totally ordered set $\{E_1, \ldots, E_s\}$ of exceptional objects is a ***full exceptional collection*** if

- Each E_i is A_i-exceptional for $A_i = \mathrm{End}(E_i)$
- $\mathrm{Hom}(E_i, E_j[n]) = 0$ for all n, whenever $i > j$
- The collection of E_i generate the category

Observation. A (triangulated) category can be decomposed as the derived categories of (smooth) points if and only if it possesses a full étale exceptional collection.

Question. Which varieties admit full exceptional collections?

Theorem. (*Ballard, Duncan, McFadden, 2017*)
A k-variety X admits a full exceptional collection if and only if $X_{k^{sep}}$ admits a full exceptional collection whose components are permuted by the action of Gal(k^{sep}/k).

Theorem. (*Kawamata, 2006/2013*) For a smooth toric variety X over an algebraically closed field k of characteristic zero, $D^b(X)$ admits a full exceptional collection.

In general, determining if $D^b(X)$ admits a full exceptional collection for an arbitrary variety X is difficult.

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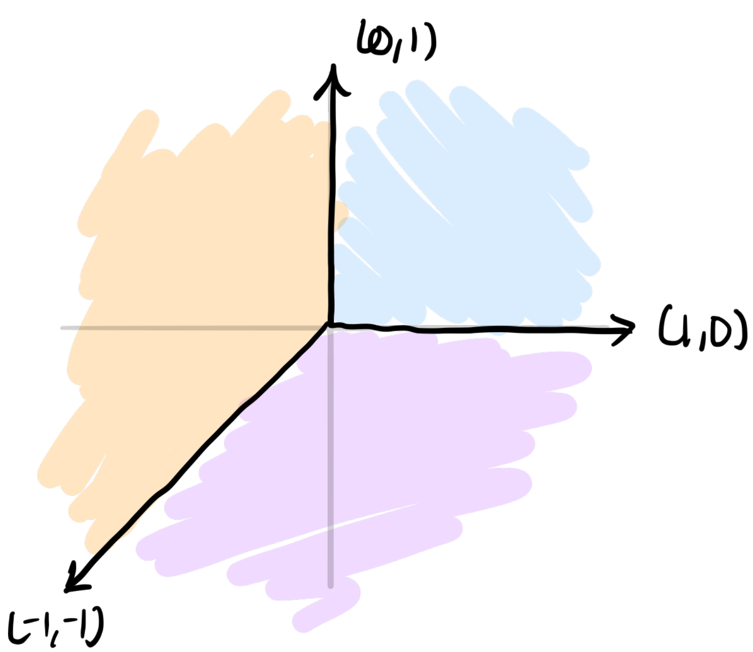
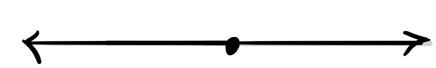
<u>Definition.</u> A ***torus*** over a field k is an algebraic group T such that $T_{k^{sep}} \cong \mathbb{G}_m^n$. We say that T is ***split*** if $T \cong \mathbb{G}_m^n$.

<u>Definition.</u> A ***toric variety*** is an algebraic variety that contains an algebraic torus as a dense open subset in such a way that the action of the torus on itself extends to the entire variety.

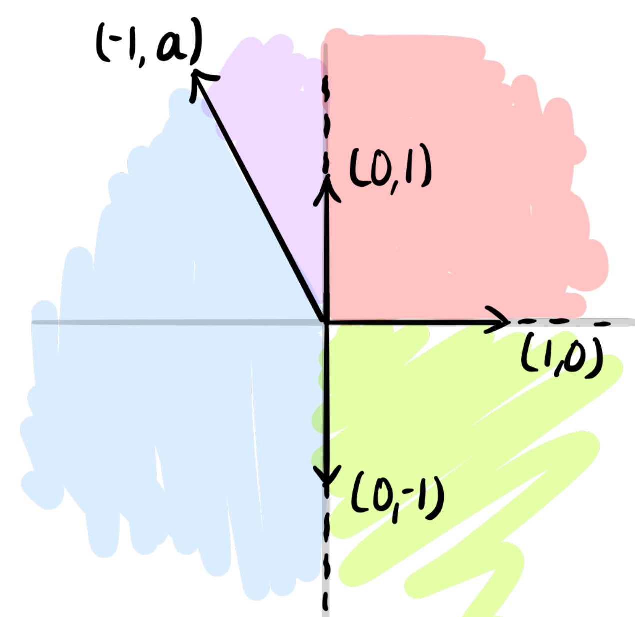
The geometry of a normal toric variety over a separably closed field is completely determined by the combinatorial data of its associated fan- a diagram that describes how the variety is constructed via the gluing of torus orbits.



$\mathbb{P}^2$



$\mathbb{P}^1$

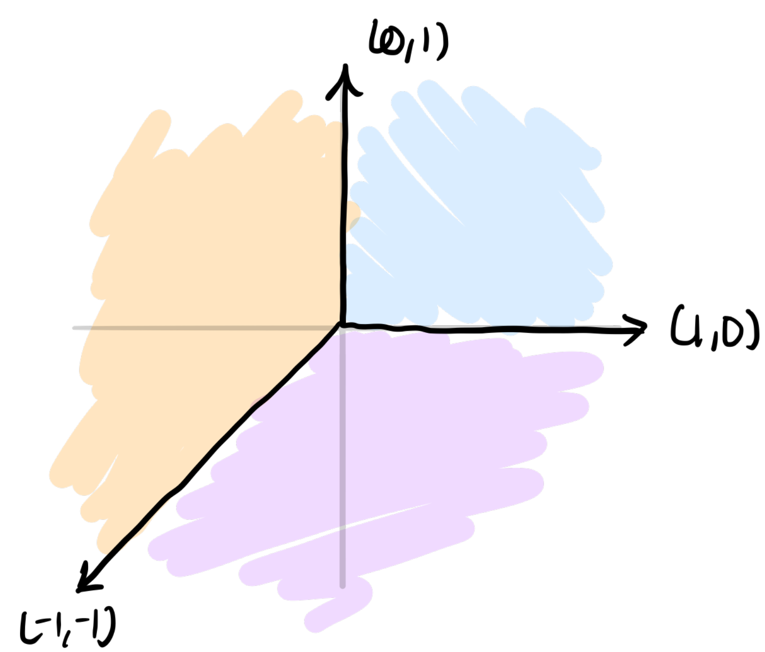


$\mathscr{H}_a$

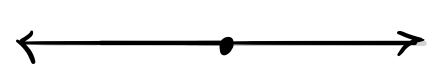
<u>Definition.</u> An ***arithmetic toric variety*** over k is a variety that base-changes to a toric variety over k^{sep}.

“Twisted Forms of Toric Varieties” ([*Duncan. 2014*](#))

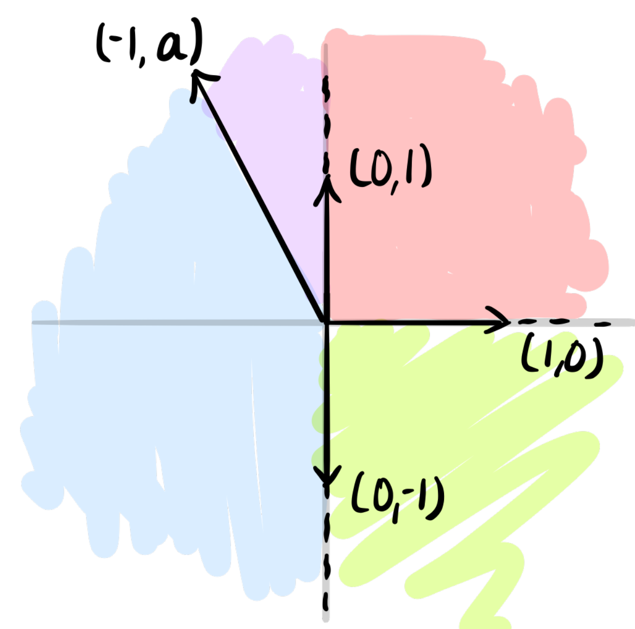
“Arithmetic Toric Varieties” ([*Elizondo, Lima-Filho, Sottile, Teitler. 2010*](#))



$\mathbb{P}^2$



$\mathbb{P}^1$



$\mathscr{H}_a$

Example

$\mathbb{P}^1_{\mathbb{R}}$

$\mathbb{R}^\times$

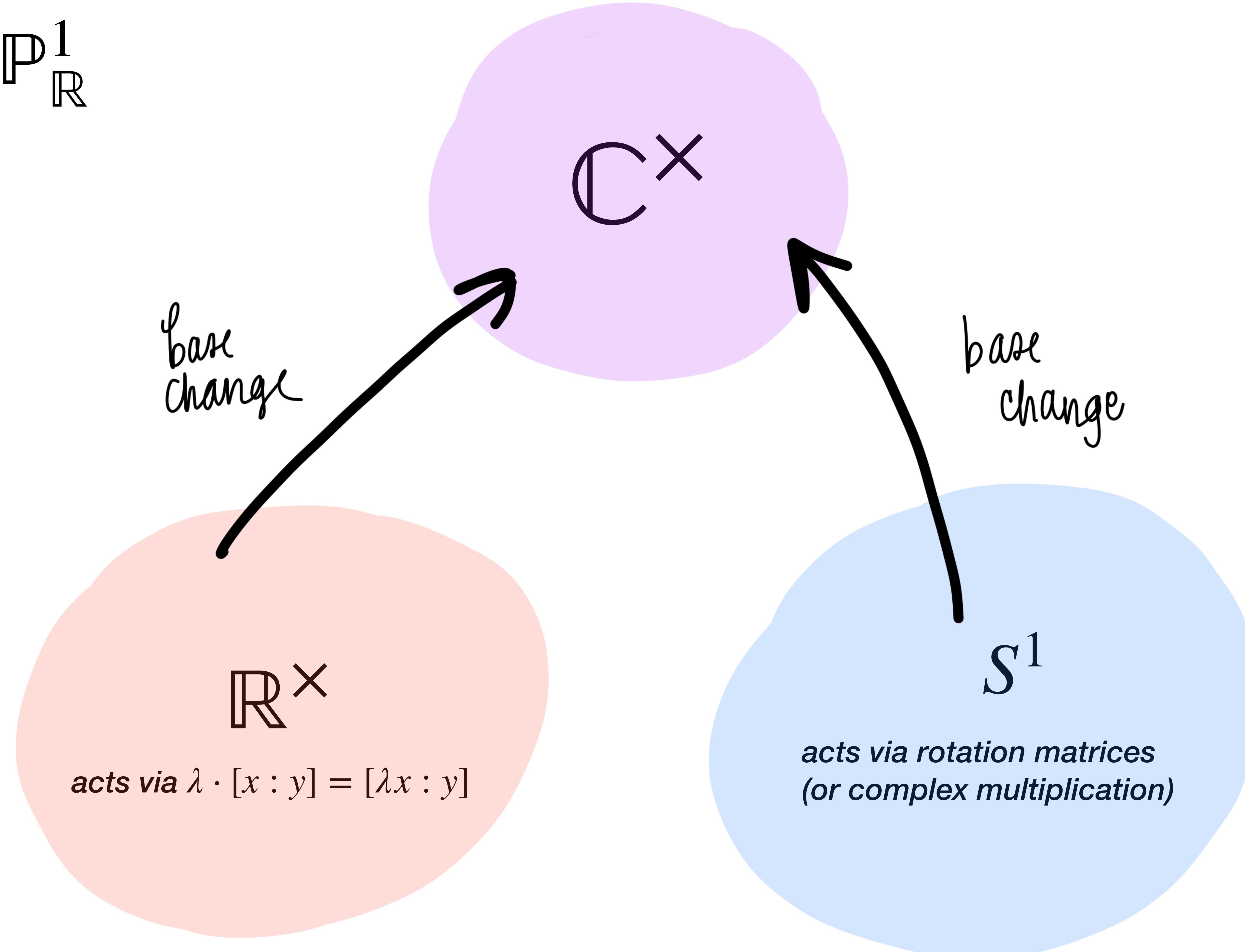
acts via $\lambda \cdot [x : y] = [\lambda x : y]$

S^1

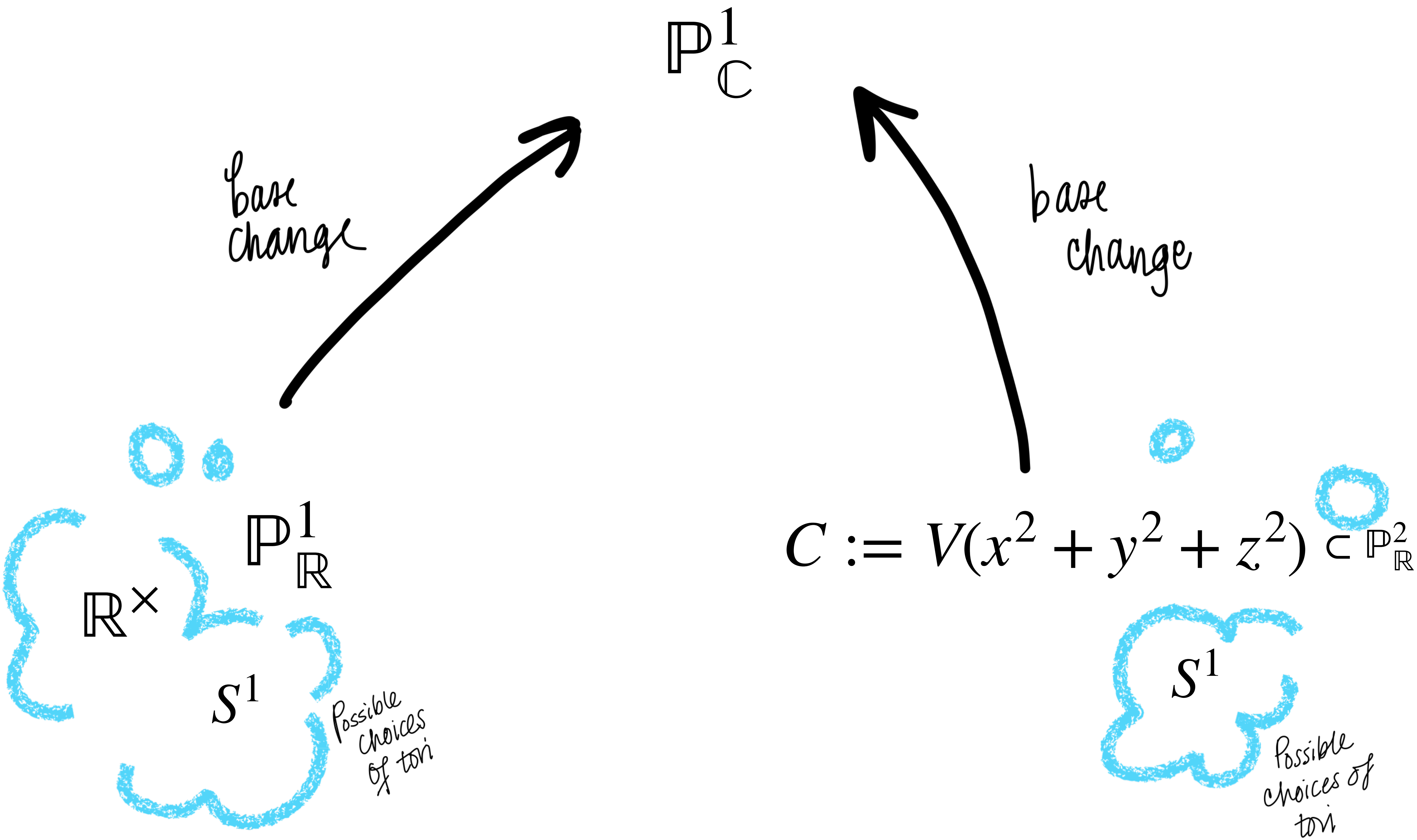
acts via rotation matrices
(or complex multiplication)

Example

$\mathbb{P}^1_{\mathbb{R}}$



Example:

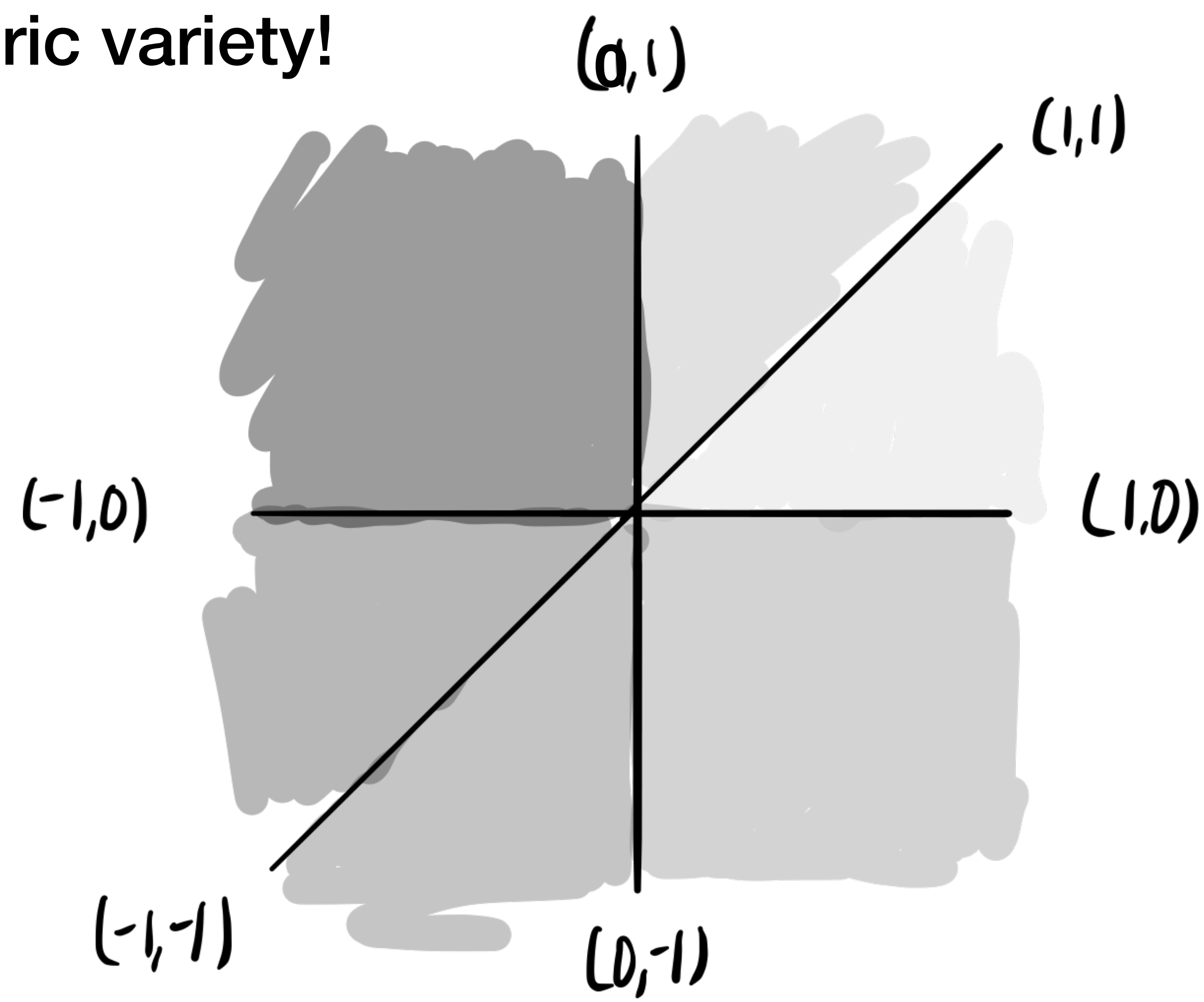

$\mathbb{P}^1_{\mathbb{C}}$
base change
base change
$\mathbb{P}^1_{\mathbb{R}}$
$C := V(x^2 + y^2 + z^2) \subset \mathbb{P}^2_{\mathbb{R}}$
$\mathbb{R}^\times$
S^1
Possible choices of tori
S^1
Possible choices of tori

Theorem. (*Ballard, L.*) If X is a generalized Del Pezzo variety, then X has a rational point if and only if $D^b(X)$ admits a full étale exceptional collection.

an arithmetic toric variety!

The fan for V_{2n} has the following rays:

$$e_0 = (-1,\ldots,-1) \quad \overline{e_0} = (1,\ldots,1)$$
$$e_1 = (1,0,\ldots,0) \quad \overline{e_1} = (-1,0,\ldots,0)$$
$$\vdots \qquad \vdots$$
$$e_n = (0,0,\ldots,1) \quad \overline{e_n} = (0,0,\ldots,-1)$$



V_2 = *Del Pezzo surface of degree 6 (over k^{sep})*

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Theorem. (*Castravet, Tevelev 2017*) $D^b(V_{2n})$ admits a full exceptional collection that is stable under the action of Gal(k^{sep}/k).

Theorem. (*Klyachko, Voskresenkii 1984*) An arithmetic toric variety has a rational point whenever the T-torsor has a rational point. Further, a T-torsor has a rational point if and only if it is the trivial T-torsor.

to find rational points on a toric variety, we just need to detect when the torsor is trivial.

Theorem. (*Ballard, L.*) If X is a generalized Del Pezzo variety, then X has a rational point if and only if $D^b(X)$ admits a full étale exceptional collection.

*For forms of dP6, **M. Blunk** showed this in 2008!*

Example application:

$\text{End}(\mathscr{O}) = \mathbb{C}$

$\text{End}(\mathscr{O}(1)) = \mathbb{C}$

$$D^b\left(\mathbb{P}^1_{\mathbb{C}}\right) = \langle \mathscr{O}, \mathscr{O}(1) \rangle$$

$$D^b(C) = \langle \mathscr{O}, \mathscr{F} \rangle$$

$\mathscr{F} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathscr{O}(1)^{\oplus 2}$

$\text{End}(\mathscr{F}) = \mathbb{H}$

$\mathbb{H}$ is not isomorphic to $\mathbb{R}$, nor is $\mathbb{H}$ a separable extension of $\mathbb{R}$

conclusion: C is not rational

Proof sketch:

Let F be our base field, with V_{2n} defined over F^{sep}.

$\{K, L\}$ ⟷ $\left\{ \begin{array}{l} \text{Choice of tori of} \\ F - \text{form of } V_{2n} \end{array} \right\}$

(Étale $F-$algebras of rank 2 and rank $n+1$ respectively)

- Fixing F-algebras K and L of appropriate rank gives a T-toric variety X which is an F-form of V_{2n}.
- We can construct Brauer classes $B \in \mathrm{Br}(K)$ and $Q \in \mathrm{Br}(L)$ that 'detect' when the T-torsor is trivial: B, Q are split if and only if T-torsor is trivial. (***Klyachko & Voskresenskii***: trivial T-torsor => F-rational point)
- V_{2n} has a Galois-stable full exceptional collection $D^b(V_{2n}) = \langle \mathbb{E}_1, \ldots, \mathbb{E}_n \rangle$ (due to ***Ballard, Duncan, & McFaddin***) where each $\mathbb{E}_k$ descends to an object $E_k \in D^b(X)$.
- One can show that there exist i, j such that $\left[\mathrm{End}_X(E_i)\right] \sim [B]$ and $\left[\mathrm{End}_X(E_j)\right] \sim [Q]$

Proof sketch:

- An exceptional collection for $D^b(X)$ yields a decomposition

$$\mathscr{U}(X) \simeq \bigoplus_i \mathscr{U}(D_i)$$

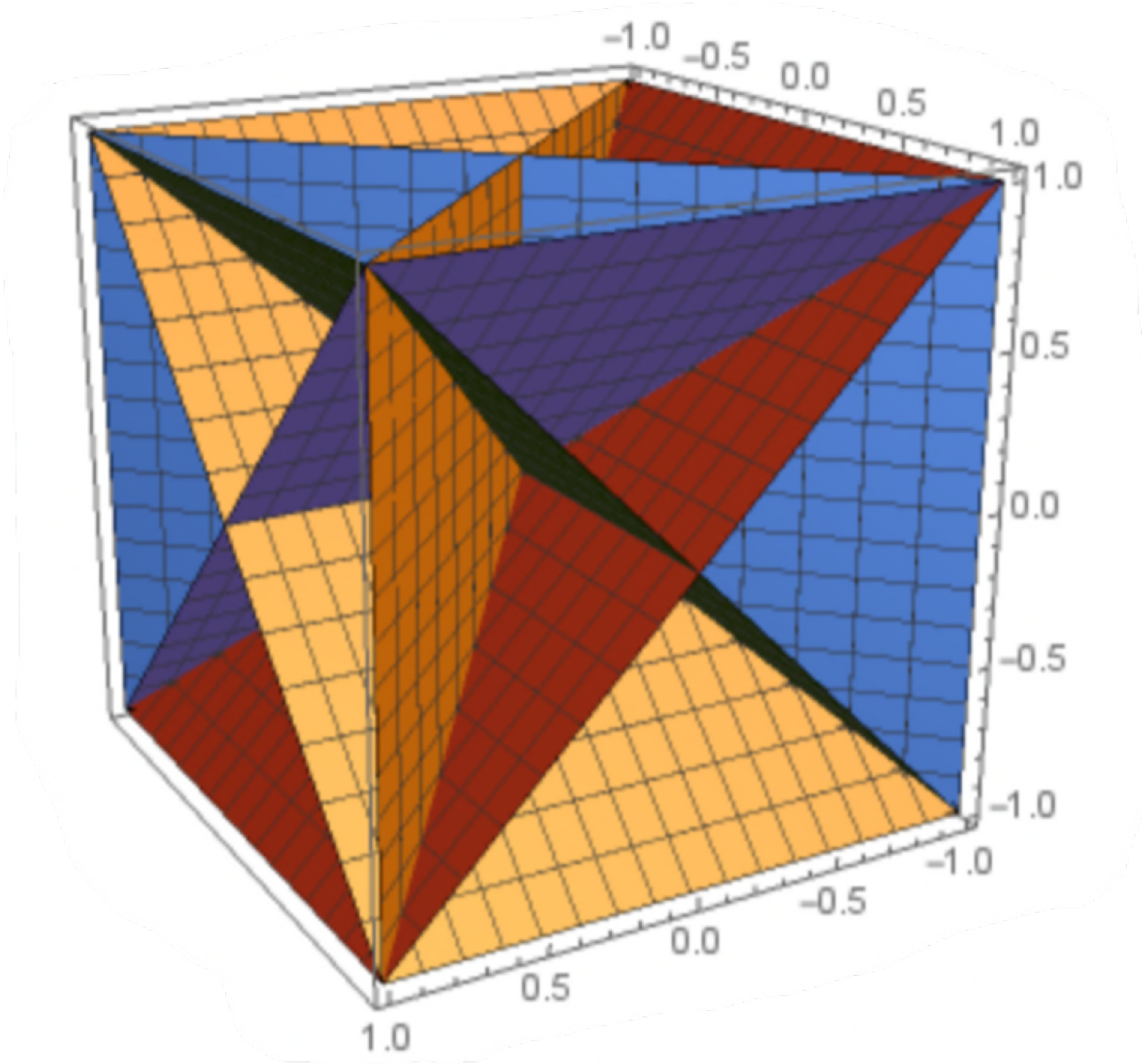
of its corresponding *universal additive invariant* with $D_i = \text{End}(E_i)$

Question:

Can a similar statement for general arithmetic toric varieties be made?

is a full étale exceptional collection a strong enough condition to guarantee the existence of a rational point?

Theorem. (*Ballard, Duncan, L., McFaddin*) There exists a smooth and geometrically irreducible threefold over $\mathbb{Q}$ with no $\mathbb{Q}$-points but whose derived category admits a full étale exceptional collection.



Thank you!